REAL MATHEMATICS

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OXFORD TECHNICAL PUBLICATIONS

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INTENDED MAINLY FOR PRACTICAL ENGINEERS, AS AN AID TO THE STUDY AND COMPREHENSION OF MATHEMATICS

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PREFACE

One of the principal objects of this book is to offer assistance to practical engineers and engineering students in the acquisition of a real, serviceable and sound mathematical equipment.

The book is intended to augment the standard textbooks and orthodox methods of study. Here and there modifications of the views in common acceptance are suggested; but there is no desire to supplant or alter them, except where fallacies are exposed—as, for example, in the extraction of roots, and in Imaginary Quantities. The desire is much more to bring about a change of attitude towards mathematics than to propose methods which shall merely be different from those in use—to show the thing as an actual, tangible reality, instead of as a collection of rigid and unrelated rules and formulae; and to persuade students to touch and handle it for themselves with confidence and understanding, instead of regarding it from afar as some rather awful and totally incomprehensible abstraction. It is felt that there is pressing need for the adoption, by all concerned, of a different standpoint from which mathematics may be regarded by those who require it and depend upon it as a part of their working equipment; so that the subject may be studied and practised throughout in the light of simple and straightforward human reality.

Such an attitude as that suggested in the following pages should prove useful to practical engineers and those

concerned with constructional work of all kinds; and perhaps not less so to those whose dealings with mathematics are of a more academic nature. It permits (as the ordinary human mind demands) the visualisation—actual or mental—of the various states and changes which are implied by the operations and processes employed in mathematical calculation and investigation; and thus brings mathematics within the grasp of every normal, average human being—for the man-in-the-street can understand the realities of common life as well as the most learned professor, if those realities be presented to him in physical form. It effects an enormous saving in time; for the fundamental principles of physics and mechanics—indeed, of natural philosophy in its widest aspects—form the merchandise with which the mathematical work deals as a kind of accountancy. Moreover, it tends to the minimisation of error in mathematical work itself; to reveal the causes of (and therefore to reconcile) the too numerous discrepancies between the inferences drawn from mathematical reasoning and the conclusions ascertained from observed fact; and to prevent recurrence of such discrepancies in future research.

It is, unfortunately, a fact that students in general regard mathematics as a mere school-subject, the development of which may enable one to solve weird puzzles; but which has no bearing whatever upon the affairs of everyday life. The subject is studied, mostly, as a collection of unrelated rules and tricks; and because there is no chain of connected reasoning in terms of real things, it is regarded with suspicion, distrust and active dislike.

From a fairly extensive and intimate acquaintance with engineering students the Author has found that the things which seem difficult and elusive to them are the very same things which, a few years ago, seemed difficult and elusive to him. The treatment here presented has proved as helpful to a large number of them as it did to him.

By some means—which, perhaps, we need not investi-

gate too closely—the physical realities of mathematics have become swathed about with wrappings of mystery and suggestions of the supernatural. In the following pages an attempt is made to remove some of those wrappings, so that the truth may stand revealed in the light of day; and, as will be seen, the invariable result is that what appeared to be a highly complex and incomprehensible business turns out to be a quite plain and homely affair, which any one who wishes may understand and use. More important still, it is clearly the work of ordinary men seeking, by human means, solutions for the problems of ordinary human life.

As a very superficial indication of the results to which this treatment leads, it is shown that (to cite only three instances) cases of algebraical factorisation which are usually considered difficult may be actually performed in a few minutes with counters or other convenient symbol-things on a Halma board; quadratic equations may be solved by similar means; and the processes of differentiation and integration may be watched in actual operation.

If the method of presentation seems less concise than that of the text-books, the reason is that the endeavour has been to give a complete argument for each of the few cases considered. It is often contended that ultra-conciseness is necessary because, even with wholesale abbreviation of the underlying arguments, the average student takes a long time to acquire a working knowledge of mathematics —which contention has always seemed to the Author about as reasonable as would be the suggestion that, because a man fails to become rapidly an efficient rider when provided with half a horse, he would therefore take much longer to acquire the art if provided with a complete and serviceable animal. Half an argument is no less dead and useless a thing than is half a horse—indeed, perhaps it is more so.

Moreover, the fact that studying mathematics generally resolves itself into the working of countless examples and exercises of a totally abstract or unpractical character,

gives many students a dislike for the subject. It is the old trouble—the glorification of technique into an end, instead of its utilisation as a means to an end ; the insisting upon technique for its own sake, instead of its proper cultivation as a means of expression.

So far as the Author is aware, no such treatment of mathematics has been published previously ; and for this reason the scope of the present volume has been confined to such branches of the work as are most useful to the ordinary engineer and student. Much more is available if required.

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CHAPTER I

INTRODUCTION

Need for Understanding.—Every normal man desires to understand the things he uses; and if he be required to use an implement which he does not understand, he will use it with more or less suspicion and distrust. In such circumstances, a man feels that he is a mere operator, pressing buttons and pulling levers which may or may not set in motion the machinery which they are supposed to control; and if, after operating satisfactorily ninety-nine times, the buttons and levers fail him at the hundredth, it is no more than he expected. Self-respect—that vital necessity to a good engineer—is impossible under such conditions.

Now, a man may understand the action and workings of a mechanism without being himself able to make such a mechanism with all the polished craftsmanship necessary for a marketable article; but given such understanding, he will be able to make a rough working model or a drawing of the mechanism, tracing the functions of the various parts throughout, and this is all that is necessary in most cases. Many arguments have been put forward both for and against requiring that a man shall be able to *make* a thing before he is allowed to use it; and too often it would seem that enthusiasm for the view held has been allowed to obscure the vision—to foster a desire for a plausible argument rather than one which is strictly truthful—with the consequence that the issue has become

vague and indefinite. For instance, one (who was unquestionably sincere in his desire that engineering students should acquire good mathematical equipment) has written that a man can tell the time correctly from a clock without being able to make the clock; which statement is both true and fallacious. A man does not use a *clock* to tell the time; he uses merely the *dial*—the circle of hour-numbers and minute-marks, and the two "hands"—and these he could make roughly for himself. As to how the hands are made to turn in accordance with the apparent altitudes of the sun, the man who gauges the passage of time by reference to a clock-face neither knows nor cares. The fact that he uses a clock-face is in itself proof that he works according to "hours" rather than to the relative positions of the earth (or of bodies on the earth) and the sun; but navigators and outdoor workers—to whom the apparent movements of the sun relatively to portions of the earth's surface are more important than conventional "hours"—gauge the passage of time by the sun rather than by a clock-face, and they understand the apparatus which they use—indeed, it is often devised by themselves—for this purpose. Similar instances in other directions, with equally simple explanations, will doubtless suggest themselves readily.

The subject is both interesting and important from many points of view besides that of education; but this is not a suitable occasion for exhaustive discussion of such matters. From broad observation it would appear safe and reasonable to conclude that if a man is to use an implement or a process with that true efficacy which is born of complete confidence in his power to command it, he must be allowed to understand its workings thoroughly. Man fears and distrusts only those things which he does not understand sufficiently to give him control of them; and this fact is too often turned to profitable account by the unscrupulous.

Mathematics and Accountancy.—No argument is neces-

sary to prove that a good knowledge and understanding of accountancy is essential to a man who would become a successful and efficient trader; and it is equally true—indeed, it is almost the same statement—that a good knowledge and understanding of mathematics is essential to a man who would become a really practical engineer.

A merchant who lacked an adequate knowledge of accountancy would be unable to record his transactions in such a form as to show clearly the state of his affairs at any moment; while he would often be unable to fix a proper relation between his selling prices and buying prices to ensure a specified margin for contingencies and profit on his whole turnover. Moreover—and this is vastly important—the activities and classified records of others working on similar or allied lines would be unintelligible to him.

The analogy between this case and that of an engineer lacking adequate mathematical equipment will be obvious.

It is quite true that many a successful trader (*i.e.* a man who has himself traded merchandise successfully, and not one of those who merely amass fortunes out of the flesh and blood of their fellow-creatures) has little knowledge of accountancy as the professional accountant knows it; but his power to dispense with such knowledge is due to the fact that his transactions were of such a nature that they could be recorded easily on some simple plan of his own devising. And as a rule it would be found that while he needed only a little accountancy, he knew that little so thoroughly that each entry in his books represented to him an actual and living reality. But even so, it is highly probable that a more comprehensive knowledge of accountancy as used by others might have enabled him to broaden his scope, by adding the results of their experience to his own, and thus to have attained a higher degree of success and efficiency, with consequent advantage both to himself and his fellows.

It is equally true that a practical engineer needs but

little mathematics for the purposes of his actual work; but he must know that little so thoroughly that every part of it is a living reality to him. Also, like the trader, he may often devise his own simple methods of recording and reckoning to suit his own purposes in dealing with the design and construction of machinery or structures; but if he use methods of his own to the exclusion of those employed by his fellows, their work and results will be to a large extent unintelligible to him—which means that he will be blind to much that may be of vital importance to him and his work.

One of the main objects of this book is to show, openly and straightforwardly, that mathematics is to the engineer precisely what accountancy is to the honest trader. It is our means for recording, clearly and concisely, states and changes in the things with which we deal, in such a manner as to provide reliable information upon which further transactions may be based; and it enables us to understand the records of others who are or have been engaged upon similar or allied work.

The manner in which mathematical expressions are commonly written, using symbolical characters and signs, is merely a system of shorthand, upon which men have agreed as a means for saving time and trouble, both in writing and for reference. There is nothing in this that need bother any ordinary man if he will approach the system in the right spirit—a spirit of frank and sportsmanlike friendliness—regarding it always as something which is offered for his use and convenience, and not as something purposely devised to be obscure and confusing. Here and there improvement might be effected by altering some symbol which is in frequent and common use, so that the symbol may more clearly indicate the reality which it represents; and any system of shorthand usable in conjunction with a living language must permit of expansion and improvement as required to meet the needs of increasing experience on the part of the people who speak the lan-

guage. At the same time, however, symbols in mathematical expressions are very like the words of a language. They are merely labels which represent, by more or less common consent, certain realities; and although, for avoiding confusion, it is desirable that words and other symbols shall be as distinctive and indicative as possible, it matters little what we call a thing provided the thing itself be commonly known and understood, and provided the name or symbol chosen to represent it be commonly accepted or commonly acceptable. It is more profitable (in the best sense of the word) to concentrate effort upon the endeavour to understand the realities than to waste it in quarrelling about mere names for them.

Attitude towards Mathematics.—Another of the main objects—and perhaps the chief of them—of this book is to show that there is nothing in the mathematics of engineering which cannot be thoroughly understood by the ordinary man, if he will take up the proper attitude towards the subject, and give it the necessary thought and practice. Naturally, the progress a man may make, and the degree of control which he may acquire, will depend upon the amount of properly directed effort which he is prepared to devote, and the sincerity with which he pursues his task; but a good mathematical equipment is within the grasp of every engineering student who will stretch out his hand to take it. Judging from the inclinations of boys in general (and girls too, for that matter) it is probably not necessary even to particularise those who become regular engineering students; and it would be difficult to name a calling or pursuit for which a man or woman would not be immeasurably better equipped with some knowledge of mathematics than without it. Such points, however, we need not discuss; it is sufficient for our present purpose to assert positively, and to demonstrate as well as we may, that any man who can earn his living in an engineering workshop or office, or out on works of construction, can have as complete a comprehension and control of mathe-

matics as he chooses to acquire, granted the prerequisites of attitude and concentration, which are easily within the power of all such.

The attitude he must adopt towards mathematics is similar to that which he adopts towards ordinary language and drawing. These are to him means of expression by which he is able to communicate with his fellows; and mathematics is another—and an immensely powerful—means of expression. An engineer does not confine language and drawing to his engineering work. He takes them with him into the ordinary affairs and pursuits of his everyday life, thus acquiring greater facility and dexterity in their use; which, in turn, enables him to express himself more effectively in his engineering work. For example, ask an engineer to describe something, quite apart from engineering as the term is commonly understood—a batting stroke with which he proposes to play a particular kind of ball at cricket; the circumstances of a street accident which he has witnessed; or his pet method for pruning a rose tree—and he will immediately produce from his pocket a pencil and paper, with which he will proceed to sketch the thing in detail—plans, elevations and sections, all complete, with figured dimensions—in amplification of his verbal description. Language and sketching are verily parts of the man's being; and if debarred from using either of them, he would often be at a loss for means to express even his simplest ideas or to record even his most ordinary impressions and experiences. Sometimes the one serves where the other fails; but generally they are more or less interwoven, and are used together, the one amplifying the other.

Now, mathematics, rightly regarded, may form another means of expression; sometimes capable of serving where all others fail, but mostly fitting in with them to give a more accurate account of some idea or impression than would be possible otherwise.

The mental and personal progress of a man depends

largely upon the means of expression at his command; for though it is true that means for expressing ideas are of little value where there are no ideas to express, it is equally true that the idea-producing faculty becomes dulled and atrophied if the ideas produced cannot be properly expressed; while the possession of means for the expression of his ideas will be instrumental in breeding other ideas. The danger lies in attempting to cultivate either faculty—that of idea-production, or that of idea-presentment—to the exclusion or detriment of the other; but for the practical engineer, living (as he does) among realities with apparently limitless possibilities of combination, it may be that the danger of over-cultivating the faculty of idea-presentment is less than for others who are concerned with less tangible things.

It must be remembered always that means for presenting or expressing ideas constitute what men call "technique"; and technique must always be a means to an end—never an end in itself. Technique is good if it makes for the better presentment of an idea or impression; while it is bad if it obscures or hinders expression. Its sole function and purpose is to facilitate and improve expression; and it can be judged properly upon no other basis than according to its success in fulfilling this function. Man should be the master of technique—not (as he too often is) its slave. The ability to solve a differential equation is, of itself, not worth five seconds of effort to acquire; but if such ability enable a man to design machines or structures more economically, or if it serve him as a key to the recorded experience of others, its value would clearly be so enormous as to lie beyond the scope of ordinary means for estimation.

The study and practice of mathematics should be taken into the very being of a man, and made actually a part of himself. In return for his trouble he will obtain such satisfaction as might be dreamed of, but could not be secured by any other means. And so far from being tedious or dull to study, it will be found of all things the most fascinating if rightly regarded—indeed, how should it be otherwise, when

it provides a medium in which a living, thinking man may express both his simplest and most complex ideas ; a medium at once more lucid and more subtle than any ordinary language, and both more broadly general and more precisely specific than any form of drawing.

Let no man think that mathematics is rigid, inert or circumscribed. Being a means of expression for living men, it is (and must always be) liquid and elastic ; it has as much latent vitality as art or music ; and it is being extended every minute as thinking men use it to express their original thoughts, and to record the findings of their original researches. It lies ready to the hand of every man, for him to use and develop according to his own particular needs, and to stamp with his own personality.

Method of Study.—To the student who would obtain a sound mathematical equipment which will serve him equally in the office, the workshop, the field, the affairs and pursuits of everyday life and recreation, and the examination room, the method of study here described is recommended.

In all branches of the work, and in all stages of progress, let the realities be kept clearly in view ; and let the process of argument be always from the realities to the mathematical expression of them—never from the mathematical expression to the realities. This will be found actually to save a great deal of time after the first breaking of the ground ; while it gives an understanding of the work such as cannot be obtained by any other means.

Let no process, operation or rule be accepted simply because it has been printed and reprinted in hundreds of books, and seems to give an answer for which examiners will allow full marks in an examination. All the operations and processes in general use should be dissected and analysed until their physical significance, their reality and their legitimacy have been demonstrated beyond the possibility of doubt, and until they are fully comprehended in all their bearings. When, in an original calculation or investigation, it appears that an expression may be

simplified by a rearrangement of the terms, the isolation of some common factor or the introduction of some variant, or any other of the methods by which expressions may be simplified, let it be ascertained before the suggested change in the expression is made, that it represents a physical operation upon the real things to which the expression relates, and that such an operation upon them is properly permissible and justifiable.

This may seem to be a large undertaking, and to savour of microscopical precision; but as a fact, it will be found an extremely simple matter after the first few attempts, while—so far from being microscopical—it gives a breadth of view which will frequently reduce to a few lines what would otherwise involve a lengthy investigation. Indeed, it will often be found, when properly cultivated, to give that desirable faculty for arriving at practical results mentally and with great rapidity—a faculty which, for all its enormous value, may be acquired in generous measure by any one who will seek it with a little diligence, yet which inspires in the minds of the uninitiated such wonder that they regard its possessor as a kind of mental Spring-heeled Jack. Seeing that man's natural inclination is always towards the rapid deduction of inference or conclusion from simple and ascertained fact, it follows that the faculty referred to above is based upon a process far more normal and instinctive than the blind groping for a formal "answer" by the automatic operation of some set rule from which all association with real things has been abstracted—and this is probably the reason why it seems so desirable to those who have been prevailed upon to work along less normal lines.

Patience is necessary, and perseverance also; but such virtues are essential characteristics of the engineer. Other things are necessary, too, such as imagination, open-mindedness, readiness to give freely without looking for reward—in fact, all the things which go to the building up of an esteemed and valued friendship; and perhaps this may afford a clue to the spirit in which mathematics should

be regarded if one would obtain the best results from it—a spirit of simple and unaffected sportsmanlike friendship, taking it into one's ordinary life and pursuits, instead of setting it apart as a cold, unresponsive thing to be confined rigorously to the classroom, the office or the study. By this means, the view obtained will be outwards instead of inwards ; and Rules—in so far as they may be helpful—will crystallise themselves from knowledge of the realities, instead of forming a collection of unrelated tricks.

Let no man think that mathematics is or can be set out completely in a book—or in a thousand books. All that can be reduced to a form suitable for printing in books is the mere husk and shell of the thing. The real thing itself lies in the mind of each man, and it is for him to develop and cultivate it to suit his needs, as an athlete or a manual labourer develops his muscles. This book, for instance—though it is offered in a sincere desire to help those who (like its author) have found mathematics elusive, mysterious and difficult to study on the orthodox lines—does not pretend to contain more than a few broad suggestions for tackling some of the most common difficulties. The actual work of perception and comprehension must be done by the student himself ; for it is simply the development of his own mind and personality, and, obviously, that cannot be done for him by any one else.

No exercises or examples are given here for practice. An abundant supply may be found in any of the good, standard school-books on mathematics. To gain confidence and control, the student should take a sufficient number of such examples and work them out fully, always with reference to real things, on the lines indicated in the following pages. But the best progress will undoubtedly be made by working in this manner upon the practical calculations of ordinary commercial engineering ; for in these a man does not need to imagine as real the things with which he is dealing. They are already real and tangible—things which he understands, and knows to be important.

If some parts of the work—particularly in the early chapters—seem too simple to need detailed discussion and analysis, let it be remembered that a good foundation is of prime importance in the erection of a stable and permanent structure. In these simple matters lies often the key to later work which appears difficult largely or solely because it is studied apart from them.

The Object of Mathematics.—Now let us look more closely into the matter, in an endeavour to see what is the real object of mathematics, and the reasons why a working control of it is so important.

From the earliest times, and in even his rudest states, man has needed means for counting things in groups. For self-preservation, he sought to know how many of his enemies were concerned in an attack upon him; how many minutes or days were available for preparation to meet a threatened assault; for how many weeks or hours a known stock of provisions would last him in the event of supplies being cut off; and manifold other numberings in which accuracy or error meant for him life or death. For the purposes of commerce, if a man could not correctly estimate the smallest number of measures of wheat for which an ox or a camel could be bartered fairly, he would doubtless soon be ruined.

We need not here examine in detail the various methods of counting which seem to have been employed by our early ancestors; though the methods themselves, the ways in which they were employed and the changes effected to meet altered needs and circumstances form a most interesting and illuminating study. It is sufficient to observe that the main object remains, in principle, exactly the same to-day as it always was; and that the methods have only been extended to deal with the more complicated "numberings" and the more complex "things" which have been brought about by communal life and progress in civilisation.

Mathematics, from the most elementary to the most advanced work, is concerned with one root principle only

—*the counting of real things in groups.* It deals with *states* and *changes* in groups or assemblages of real things under consideration, as regards the numbers of things comprised. Sometimes the "unit thing" is just *one* of the ordinary things of everyday life—such as a brick, or a complete rotation of a wheel about its axis; sometimes the "unit thing" is a fraction of some other thing commonly regarded as a unit—as when a limit of tolerance is expressed in thousandths of an inch; and sometimes, again, the "unit thing" is in itself a group or assemblage of other things—as when a consignment of slates is invoiced in "milles." But these are merely matters of convenience, the object being always to save the trouble of dealing with either an awkwardly large number of relatively small things or a microscopical fraction of a relatively large thing. Further, when comparing two groups with regard to the numbers of things which they contain, one group is treated as the "unit thing"—as in ratio and proportion; and sometimes the things forming one of the groups are different in character from those forming the other group—as when we compare the distance (some number of miles) travelled by a railway train with the time (some number of hours, say) occupied in the journey, to estimate the average "rate of motion" (or velocity) in miles per hour.

Always, however, having selected the "unit thing" with which it will be most convenient to deal, the object of our calculations and investigations is to determine or to estimate some number of those "things." Whether we seek to estimate the cross-sectional area necessary for the flange of a girder which is required to support a specified load over a known span; the speed at which an engine must work, with given cylinder diameter, stroke and steam pressure, to develop a specified power; the distances subtended by observed angles at the extremities of a measured base line; or for any other of the multitudinous purposes for which mathematical calculation and investigation is employed in human dealings; we are counting real

things in groups—either that we may know how many of such things are available or necessary for some project, or that we may be able to compare one group with another as regards the number of things contained. Always, there is some practical purpose to be served; and wherever possible, the experienced man will examine and test his estimate or determination, by reference to the ascertained facts of similar cases, before working upon it or putting it forward as reliable.

From this it follows that all practical calculations must, sooner or later, come down to arithmetic, no matter how far removed from mere "numbers of things" they may appear in their intermediate stages. Hence, it behoves every engineering student to acquire the greatest possible measure of facility and dexterity in arithmetical computation; and—perhaps more important still—to keep constantly and clearly in view the fact that even the most seemingly complicated calculation or mathematical investigation is, from start to finish, arithmetical in spirit and principle. For it is unquestionable that a calculation which, based upon some number of, say, linear inches, gives some other number of linear inches as a final conclusion, must have been concerned with numbers of linear inches throughout all its stages. Numerous instances will be found in the following pages proving the truth of this contention, by means of typical engineering calculations fully worked out, dissected, analysed, and explained in detail.

Practical calculations, no matter how simple or complex they may appear, are concerned in every step and detail with physical realities alone; and there can be no mathematics apart from physical realities, any more than there could be accountancy apart from trading.

Physical Realities.—In order that there may be no avoidable vagueness or misunderstanding, it will be well to explain what we mean by a "physical reality"; for we shall frequently need to speak of such things, and (it

may appear) not always in strictly the same sense. By a "physical reality" we mean either—

(*a*) An object or an operation which exists in so far as it may be perceived by normal human beings through the action of their senses; and which is in accordance with those classified results of general human experience sometimes spoken of as "natural laws"; *or*

(*b*) An object or an operation which, although it does not so exist, could be brought into such existence and conformity with natural laws by ordinary human agency, with no greater difficulty than is attendant upon all human projects and undertakings.

Suppose that an architect prepares a design for a building which he purposes erecting upon a vacant site; and that he makes finished drawings of the building before any preparations for constructing it are commenced. He will tell you that the height of the roof is to be so many feet above ground level; the cubical extent so many cubic feet; the number of bricks required so many thousands; and so on, though the "building" of which he speaks is represented by nothing more substantial than a few lines and washes of colour on drawing paper.

Now, it is true that the objects and operations involved in such a case do not actually exist in the hard-and-fast sense of the term; but there is no insuperable bar to their being brought into existence by ordinary human agency if the design be well conceived; and therefore we may think of them as physical realities, subject to natural laws. We could, of course, think of them as "imaginary" (seeing that they are so far only imagined); and it might be possible to arrive at some acceptable denomination—such as "realisable imaginaries"—for them. This, however, would probably lead to confusion, for the word "imaginary" has been used largely by mathematicians to denote imaginings which could never be brought into existence or conformity with natural laws by ordinary human agency.

Symbols.—Another point which should be clearly understood and fully appreciated is the nature and purpose of symbols, and their relation to physical realities.

A " symbol " is some readily obtainable and convenient thing (such as, for instance, a mark made upon a sheet of paper) which is allowed to represent or stand for some other thing, about which latter thing it is necessary or desirable to reason, but which would be either impossible or inconvenient to deal with as a physical basis for the reasoning. Since we are only going to count things, provided the thing chosen as a symbol be subject to the same natural laws as is the thing for which it stands, the conclusions indicated by reasoning with regard to the symbol will apply equally to the thing symbolised.

The condition regarding similar consistency with the same natural laws for the symbol and the thing symbolised will, perhaps, be clear upon consideration of the following remarks.

If the things to be counted are solid bodies which will neither cohere nor flow into one another, it would be useless to employ liquid or gaseous bodies to symbolise them; for the fluxion or diffusion which would occur between the symbolical bodies when brought together would have no counterpart in the behaviour of the bodies symbolised. If it be desired to symbolise two different things which, in the circumstances of the case, will not appreciably act and react chemically upon each other, it would clearly not be proper to employ as symbols two bodies which, similarly circumstanced, would be subject to violent chemical action and reaction between themselves. If the things forming a group or assemblage be not uniform in the essential particulars, it would be useless to symbolise them by things possessing either no perceptible difference, or varying in some manner which bears no perceptible relation to the variation in the things symbolised. And so on.

It will doubtless be noticed that herein lies both a similarity and a difference between the symbolism of mathe-

matics and that of chemistry. It may be well to leave the tracing and definition of this similarity and this difference between the two types of symbolism as a highly interesting and instructive exercise for the enthusiastic student, who will assuredly reap a rich reward from consideration of such matters.

Sometimes it will be found that we speak of the "realities" symbolised, in contradistinction from the symbolical things; whereas it may appear that the symbolical things are, at the time, more real than the things for which they stand. In all such cases it should be remembered that the object here is to focus attention with all possible clearness upon the simple and physical realities represented by the various steps of a reliable engineering calculation; and not to split hairs regarding the meanings of words. Words have been invented by man to serve his practical purposes in the expression of his ideas and impressions—not as fetters to chafe and gall him at every mental movement. A symbol may—indeed, it must—be a real thing, capable of being perceived by a normal human being, through the action of his senses; but it is masquerading, to suit our convenience, as something else, and for that reason it should be regarded as unreal during its employment as a symbol—*i.e.* so long as it is posing as something other than what it is in fact and to our knowledge.

Let each man think out these matters carefully and clearly for himself, adopting a straightforwardly rational attitude towards symbols and symbolism, so that he may obtain the highest and best assistance from their use, without permitting the reality which he is pursuing to become eclipsed or obscured by the symbol, and without falling into the error of investing the symbol with the attributes and qualities which belong to the reality alone. A large proportion of the disappointment and unhappiness experienced by human beings is directly due to this error, which is also responsible for much of the difficulty found in the study of mathematics and kindred subjects.

Measurements of Engineering.—An engineer is called upon to do as much measuring as most people; and because he depends upon it so much it is often assumed that his methods of measurement must necessarily be complicated and abstruse. As a fact, the measurements of engineering are neither more nor less complicated than are those of any other work which is concerned with human realities.

In addition to "number"—which is merely the result of counting real things—the measuring apparatus of the engineer consists (in principle at least) of three fundamental instruments:

(1) A foot-rule;
(2) A sun-dial; and
(3) A pair of scales.

A micrometer is nothing but a refinement of the foot-rule; while the protractor and thermometer are only modifications of it. A chronometer is, of course, closely allied to the sun-dial; and the pressure gauge and spring balance are but pairs of scales in forms convenient to the purposes for which they are intended.

For dealing with combinations of his fundamental "things"—Distance, Time and Force—his measurements must, of course, be combined also; but there is nothing vague or mysterious about such combinations, as is shown in Chapter XII. and elsewhere in the following pages. The same common-sense procedure is followed as would serve a greengrocer or a paperhanger. First, we decide as to what is the most convenient and suitable "thing" to adopt as our unit; and then we count the number of such things forming an assemblage, either available or required for some proposal.

After all, if a mathematical investigation does in fact lead to a reliable indication of (say) the deflection which will be produced by a specified load upon a particular beam, there cannot be any part or factor, step or process in the investigation of a supernatural character. If (other

things remaining constant) the deflection does vary as the load, the relation existing between them must be within the comprehension of the ordinary, normal men who deal with beams. And the game will be played none the worse for having all the cards plainly in view on the table.

The deflection of a beam, as estimated by mathematical investigation, is merely the *span* varied according to ratios which are determined by the facts of the case—the main variation being a reduction, according to the ratio borne by the bending moment to which the beam is actually subjected, to the full (hypothetical) moment of its elastic resistance. This may be demonstrated quite easily beyond the possibility of doubt or question; and all the calculations and mathematical investigations of practical engineering are as real and reasonable as is this one.

CHAPTER II

ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION

Addition.—Such statements as $3+5=8$ are frequently made, and are accepted without question as sufficiently complete statements of fact. There is, however, more in such a statement than may appear on the surface; and the fact that many engineers find mathematics troublesome and difficult to understand is largely due to their having failed to appreciate fully the realities which underlie the process known as addition.

Let us examine the matter more closely, in an endeavour to ascertain what is really our meaning when we make such a statement.

First, we do *not* mean that the numbers 3 and 5 added together make the number 8; for numbers have no significance of their own apart from the real "things" to which they relate. The figure 3, for example, has no particular meaning unless it refers to some real thing or things; and this is true of all numbers and symbols, as is demonstrated in Chapter XII.

It would seem, then, that the statement $3+5=8$ may mean that if 3 things be added to 5 things, the result will be 8 things in all; but even this is not sufficiently precise, for if 3 apples be added to (*i.e.* brought into juxtaposition with) 5 bananas, the result could not properly be considered as 8 unless we brought in some other name or classification sufficiently comprehensive to include an apple and a banana

under a single denomination—say, 8 "pieces of fruit." Then, however, the statement would have been altered in sense, and made less definite; because 3 melons and 5 peaches would make 8 pieces of fruit, just as well as 3 apples and 5 bananas. The statement would then mean only that 3 pieces of fruit added to 5 pieces of fruit make a total of 8 pieces of fruit; and this last observation gives us our first really definite clue—that all the things must be alike (or, at least, sufficiently so for the purposes of the statement) if the addition of 3 and 5 is to give 8. So the people who boast of their ability to add two and two without obtaining a result other than four are making a claim which they could seldom substantiate, even with real things; for of all the many and various sets of circumstances in which two things may be added to two things, there is only one typical set—that in which each of the first two is sufficiently like its companion, and both of them sufficiently like each of the second two, to justify the use of a single denomination—in which the addition will result in four.

Again, since we cannot create, we cannot add 3 things to 5 things unless we have at least 8 things at our disposal. We can, of course, employ things under our control as symbols to represent other things not under our control; and the result of an addition performed with the symbols will (provided the things symbolised be subject to the same natural laws as are the things employed as symbols) apply, for the purposes of argument, equally to the things symbolised. This is the basis of nearly all practical designing.

The real meaning of the statement $3+5=8$, then, is simply this: If there be 3 things forming a group, and 5 things elsewhere, all the things being sufficiently similar to justify the use of a single denomination or classification, we may bring the 5 things and place them together with the 3 things, making a total assemblage of 8 things. The assemblage may be of any or no particular arrangement,

orderly or irregular. All that we mean is that our attention is focussed upon it as a group, to the exclusion of all other things, for the purpose of the operation. Even the bringing together of the component subgroups may be a mental (instead of a physical) operation; a man who owns 3 sheep in England and 5 in Australia need not transport either subgroup to the other for the purpose of ascertaining the total number of sheep in his possession. He may bring them together mentally—though it is probable that, whatever method of mental visualisation he might employ for this purpose, his process would be largely or entirely based upon the use of symbols.

Now, let there be no mistake or vagueness about this point. It is this principle of the arrangement of real things to form groups or assemblages which underlies and forms the basis of mathematics. Understanding it, everything is simple, regular and logical; ignoring it, the various operations and processes seem disconnected and abstruse. All—even the most advanced—mathematical work is really *addition*, suitably modified to meet the needs of particular circumstances for numbering (or counting), as will, it is hoped, be seen from a perusal of the following pages and chapters.

It is both interesting and instructive to notice that, in the practical process of arithmetical addition, we frequently adjust the magnitude of our "unit things" to suit the needs and conveniences of the work; and also that we employ a device, which really amounts to an imaginary automatic binding machine, to effect this adjustment. Perhaps the processes will be most clearly seen from a typical example, fully worked out and analysed.

EXAMPLE I.—*Add together* 4167 *and* 935

Now, these two numbers tell us that there are 4167 things in one group and 935 things in another group, all the things in both groups being similar. We are asked to collect them all into a single assemblage, and to find the total number of things in that assemblage.

In the first number, the 7 means that there are 7 loose things; the 6 means that there are 60 things, which the imaginary automatic binding machine has parcelled up into 6 "packets," each packet containing 10 things; the 1 that there are 100 things, which the machine has parcelled up, first into 10 packets of 10 things each and then into 1 "bundle" containing 10 packets; and the 4 that there are 4000 things, which the machine has parcelled or bound up, first into 400 packets of 10 things each, then into 40 bundles of 10 packets each, and then into 4 "sheaves," each sheaf containing 10 bundles.

Similarly, with the other number, there are 5 loose things, 3 "packets" of 10, and 9 "bundles," each bundle containing 10 packets of 10 things.

We start by adding together the two sets of loose things, and as soon as 10 things are collected, the machine instantly and automatically binds them up into a "packet." Next we add the "packets," the machine instantly binding them up, if 10 or more are collected, into a "bundle," and so on.

Here the "unit things" are adjusted in magnitude, successively binding them up into "packets" of 10, "bundles" of 100, "sheaves" of 1000, and so on, the machine working on a regular basis or scale of 10. In dealing with additions of money, the adjustment would be on a scale of 12 to bind the pence up into shillings, and on a scale of 20 to bind the shillings up into pounds, after which the machine would work on a scale of 10 in binding up the pounds into "packets," "bundles," "sheaves," and so on. The adjustments and scales for other practical cases will be apparent without detailed explanation here.

The names ("packets," "bundles," etc.) here employed for the successive parcellings of the machine are put forward as suggestions only, and in no way as specific titles. It is well to adopt distinctive names for such parcellings, at least in the early stages; but each student may adopt names of his own devising if those suggested do not appeal to his imagination with sufficient clearness—indeed, it may be found helpful to do so, even if no particular objection to those suggested be felt.

Returning to the Example—the 5 and 7 loose things make 12, *i.e.* one packet of 10, and 2 *loose things* over; the 3 and 6 packets of 10, together with the one just bound up from the

previously loose things, make, in all, 10 packets, which form 1 bundle, and leave *no packets* over ; the 9 and 1 bundles, with the one newly formed from the packets of 10, make 11 bundles, forming 1 sheaf, and leaving 1 *bundle* over, and the 4 sheaves, with that just formed from the bundles, make 5 sheaves.

The total number of things resulting from the addition—*i.e.* 5 sheaves, 1 bundle, 0 packets, and 2 loose things—is, of course, written as 5102.

A further example may serve to show that the principle involved in algebraic (or " generalised ") addition is precisely that of ordinary arithmetical addition.

EXAMPLE II.—*Add together :* $2a$, $3c$, $4b$, $7a$, $5c$, $6a$, $9b$, *and* $8c$

Here, just as before, we have several separate component groups to collect into a single assemblage for counting. Also, as before, each subgroup ($2a$, $3c$, etc.) contains some number of real things, and the essential condition—that *all* the things concerned are similar—applies equally to this as to all other cases of addition. The fact that these component groups are themselves subdivided—2 subgroups, each comprising some number " a " of things ; 3 subgroups, each comprising some number " b " of things, and so on—does not constitute any material difference between this case and that of Example I., for the component groups there could have been reckoned as 9 subgroups comprising 463 things each, and 5 subgroups comprising 187 things each. The only difference between the two cases lies in the fact that the numbers of things forming the subcomponent groups of Example I. could be stated—that is to say, the case being *particular*, the things forming these subgroups could be actually counted ; whereas in Example II., the case being *typical* or *general* instead of particular, they cannot be so counted unless the conditions be particularised. Nevertheless, the symbols a, b and c in Example II. represent each some number of similar things, just as do the symbols 463 and 187 in the subdivided component groups of Example I.

The fact that several of our component groups are described as comprising numbers of subgroups all containing a things, indicates that each of these subgroups may be regarded temporarily as a " unit thing " for the purposes of a partial

addition; similarly, each subgroup containing b things may be regarded as a different "unit thing," and each of those containing c things as yet another different "unit thing."

We collect all the subgroups of a things into a single group containing $2a+7a+6a=15a$, all the subgroups of b things into one containing $4b+9b=13b$, and all those of c things into one containing $3c+5c+8c=16c$, the whole assemblage being enumerated as $15a+13b+16c$.

Before the total assemblage could be stated as some definite number of things, a, b and c would have to be particularised according to the facts or requirements of an individual case. Thus, if a, b and c were found to represent 2, 5 and 7 things respectively, the assemblage would comprise: $(15\times 2)+(13\times 5)+(16\times 7)=30+65+112=207$ things.

The question may present itself in the minds of some —and it is a good question, well worth considering—as to whether the symbols a, b and c in Example II. might not represent each a *different thing*, instead of each representing some different number of things, all those things being similar, as stated above.

Now, the sole purpose of mathematics in engineering is to serve real convenience and to effect the truest economy; and hence, if it be convenient, *in any particular case*, that symbols should be employed in this way, there is not even the slightest ground for objection. For instance, in taking account of some stores, we might use the letter a to denote "one Apple"; the letter b to denote "one Bun"; and c "one Cigarette." If the stocktaking yielded the records of Example II., the total stock would comprise, as shown, 15 apples, 13 buns and 16 cigarettes; and there are, doubtless, many kinds of cases in practical work where such procedure would be convenient and effective. But it must be observed that such use of the symbols is not permissible except in particular circumstances and for special purposes; for it is not consistent with the plain, straightforward basis on which mathematics rests, as will be readily seen on taking the argument one step farther.

Let us test the symbolism by applying it to multiplication—and whatever holds good in addition must hold equally in the process known as multiplication, for (as will be shown presently) this latter process is in fact neither more nor less than addition.

If, then, we wrote $15a \times 13b$, using the symbols a and b to denote each a different thing—say, one apple, and one bun—as suggested, what would the statement mean? Obviously, we cannot multiply 15 apples by 13 buns; while, arguing on the only lines practically acceptable—viz. that 2×7 means merely an assemblage of real and similar things, arranged (or capable of arrangement) in two rows, each row containing seven things—we should be compelled to admit that our statement carried the absurd implication of an assemblage comprising "15 apples rows," each row containing "13 buns things." It would be foolish to say that the result is "195 apple-buns," and to accept such a result without troubling to inquire into the nature of an "apple-bun," for that is merely to ignore the obstacle without overcoming it. No obstacle is encountered if we proceed on the basis of simple, natural fact; and therefore we must conclude that, for mathematical purposes in general, the symbol a, for instance, means some number of real and similar things, just as does the symbol 5; the only difference between them being that the 5 is particular, whereas the a is not. The 5 means five always; whereas the a may be 5 to-day, 3 to-morrow, and 6279 the next day, or any other number representing the facts or requirements of a particular case.

Subtraction.—Subtraction is merely the reverse of addition; and the realities which underlie addition underlie subtraction equally. Addition consists of bringing component groups of things together, and counting the things in the total assemblage. Subtraction consists of removing some of the things from an assemblage, and counting those which remain.

The statement $8-3=5$ means only that if, from a

group comprising 8 similar things, 3 of those things be removed elsewhere, 5 things will remain. Since we cannot annihilate matter any more than we can create it, the things " subtracted " from a group are merely removed and placed elsewhere. Just as in addition, we may perform the removal in imagination or by the aid of suitable symbols when it would be impracticable or inconvenient to effect it physically.

In the practical process of arithmetical subtraction, the device employed in arithmetical addition is reversed. Instead of binding up our " things," " packets," " bundles," etc., we break one open as required to permit the stipulated withdrawal. This will be seen from the following example, which will be seen to correspond with Example I.

EXAMPLE III.—*Subtract* 935 *from* 5102

We are here given a group of 5102 real and similar things, and are asked to find the number of things remaining after 935 have been removed or withdrawn from the assemblage.

There are only 2 loose things, and therefore we cannot remove 5 straight away, nor have we any " packets " to break open ; but we can break open our " bundle " of 10 packets, and one of these we can break open (giving, with the original 2, a total of 12 loose things from which to withdraw 5), leaving the other 9 unbroken. After taking away the specified 5 loose things, 7 *will remain*, and we may then take away the specified 3 " packets " from the 9 which, with that broken open, originally formed the 1 " bundle." This will leave us with 6 "*packets*." The only " bundle " we had has been broken up, and hence, to enable us to withdraw 9 bundles, as required, we must break open one of our five " sheaves," leaving four unbroken. The sheaf broken open will yield 10 bundles, and, withdrawing the stipulated 9, we are left with 1 *bundle*. No further withdrawals being required, we are left with the 4 *sheaves* which we had no occasion to break open, the reduced assemblage now comprising 4 sheaves, 1 bundle, 6 packets, and 7 loose things, which will, of course, be recorded as 4167.

The corresponding processes of breaking open pounds

into shillings, and shillings into pence, in subtracting one sum of money from another, will be clear without further explanation; as will also the breaking open of parcellings in other cases of practical subtraction.

Now, an important point arises here regarding the possibility of subtraction *beyond the total contents of a group*. In working Example III., when we were asked to withdraw 5 loose things with only 2 available, the difficulty was met by breaking open a packet. How would it be if no packet had been available to break open? To press the matter farther, how would it be if the number of things to be subtracted were in excess of the number of things forming the group from which the subtraction is to be made?

Clearly, we cannot give away things we do not possess; and therefore, strictly, subtraction becomes impossible in such circumstances. Some solution of the difficulty must be found, however, for this is a business matter, and if it were held that a man could not make payments beyond the amount of ready money in his immediate possession, business would be impossible. Purchases and commitments would be limited to cash actually in hand, the system known as "credit" (on which the business of the world is and must be based) would be barred as impossible, and an unscrupulous person would only have to hide his possessions to secure relief from his obligations.

After all, the difficulty is not a very serious one, so far as the mere task of recording is concerned; for we need only determine the amount outstanding after the withdrawal has been effected to the full extent of the available resources. In business, of course, this outstanding amount would be spoken of as a "Deficit Balance."

Suppose we were asked to subtract 1000 from 985. This would be equivalent to asking a man to pay out 1000 things when his total possessions comprised 985 of those things. Clearly, after handing over the whole 985, there would remain an undischarged debt of 15; and we

should record the result as -15, the negative sign indicating that the man's state is 15 below nothing. He must obtain and hand over those further 15 things before he can be regarded properly as possessing even nothing.

This introduces the important matter of "positive" and "negative" quantities; and as it is necessary that their significance and reality should be quite clearly understood (so that they may be handled in practical work with perfect confidence and without any possibility of confusion), we will consider this matter now.

Positive and Negative Quantities.—Let us consider a group of 8 oranges, placed upon a table for the purposes of some proposed operation. We should represent their number as 8; and no question would arise as to whether the figure 8 should be preceded by a plus sign or by a minus sign. The oranges are there, they are all real and similar, and we have counted them; and there is an end of the matter. They are neither "positive oranges" nor "negative oranges."

Now let us separate the oranges into two groups, one of 5 and the other of 3; and let us concentrate our attention upon the group of 5 as a basis for observation. These 5 oranges are, as we agreed, neither positive nor negative.

If we then brought 2 oranges from the group of 3 across the table-top into juxtaposition with the 5, we might record the change by means of the statement $5+2=7$, and the "+" in that statement would signify the *coming into the group* of further oranges. Conversely, if we again remove the same 2 oranges from the group, we may record the change by means of the statement $7-2=5$, and the "$-$" would signify the *going out of the group* of some of the oranges.

Now, the 2 oranges have not been changed in any way as regards their nature. They have only been moved. They remain real all the time; and they are neither more nor less like other oranges than they were before we moved them across the table-top. Yet the same 2 oranges are

denoted as " positive " in the one case, and as " negative " in the other.

Moreover, we must not forget that, as a matter of mere accountancy if for no other reason, the changes in the group of 3 oranges consequent upon our actions should be recorded as well as those in the group of 5. Before we could bring the 2 oranges into the 5 group, we had to remove them from the 3 group, and this we should record as $3-2=1$, the minus sign indicating withdrawal. And here we notice a vastly important and illuminating fact —the *same two oranges* are (but for the time occupied in transit from one group to the other, which is not germane to the point at issue) *simultaneously negative and positive.* They are negative as regards the 3 group, and positive as regards the 5 group.

Other examples will doubtless suggest themselves, and should be carefully considered by the student until the facts are clearly grasped. For instance, if a man sets out on a journey from London northwards to Edinburgh, each mile covered northwards is, to him, positive ; and if, after covering any particular mile northward, he found it necessary to retrace his steps southwards, the same mile which he has previously counted as positive, he would regard as negative. Moreover, any particular mile covered southwards by the man desiring to proceed from London to Edinburgh is *to him* negative ; and the same mile is simultaneously positive to another man faring southwards from Edinburgh to London at the same time by the same route.

In the case of a man journeying, we are concerned with groups of real things just as with the oranges on the table-top. Here the " things " are *miles of road* ; and in the course of the journey the miles forming the group in front of him are transferred to the group of miles behind him. It is true that the transfer is effected by movement of the man instead of the miles (whereas with the oranges on the table, the oranges were moved while the observer remained

stationary), but this does not affect our reasoning in any way.

There are three important inferences to be drawn from this consideration; and if these three inferences be thoroughly understood and appreciated, there should be no difficulty or confusion in dealing with positive and negative quantities in the practical calculations and mathematical investigations of engineering.

The first inference is that the question of positive and negative arises only when things are transferred from one group to another group.

The second inference is that the signs "+" and "–" denote only *movement* or *passage* (respectively, *into* and *out of* the group under consideration) of the things with which they are concerned; the things themselves remaining real and unchanged in nature and character throughout.

And the third inference is that things are never either *absolutely positive* or *absolutely negative*; for when they pass from one group to another, the same things are *simultaneously both positive and negative.* With regard to the group which they leave, they are negative; and with regard to the group which they enter, they are positive.

The best way to deal with positive and negative quantities in practical work is to regard positive quantities as those which increase assets, and negative quantities as those which reduce assets. Thus, "+2" means that a group under consideration is *increased by* 2; and "–2" means that the group is *reduced by* 2. If the negative quantities in an expression exceed the positive quantities, we determine the excess (by subtracting the *positive* total from the negative—or, in other words, by subtracting the amount of the assets from the total liabilities), and express it as *negative*; by which we mean that, after paying out all we have, there remains a liability of that amount still to be discharged before the assets can be regarded as nothing.

Sometimes we use the negative sign to denote distances below zero level, in contradistinction from positive denoting

distances above zero level—as in thermometer readings; sometimes to denote angles measured by clockwise turning in contradistinction from positive denoting those measured by anti-clockwise turning; and sometimes in other similar ways. But a little consideration of any such cases, on the basis of positive for assets (or *incomings*), and negative for liabilities (or *outgoings*), will show that the underlying principles are really the same as those discussed above.

And let it be remembered always that, since man cannot create material things, the incomings for one group are the outgoings for another group. We cannot add to one group without subtracting from another group—and usually the subtraction must be effected before the addition can be made possible. This fact is of enormous importance in the affairs of everyday life, as well as in mathematical work.

The "Rule of Signs."—Most people who study mathematics find trouble in dealing with the "Rule of Signs"—which states that *the product of like signs is plus, and the product of unlike signs is minus*—and usually it is swallowed whole (like a large pill) for the sake of peace and quietness. Consequently it is always cropping up in awkward circumstances, and causing trouble through its being not properly grasped.

As a fact, the matter is perfectly simple and natural if regarded from the standpoint of plain fact.

Let us consider first the simplest cases, proceeding afterwards to those which may appear more complicated.

The statement $7+(+2)$, for instance, means this. Having a group of 7 things, we bring into it (signified by the first plus) the further things denoted by the symbols within the brackets. Now, "$+2$" may be regarded, as we have seen, as denoting *two things which operate to increase assets*, and bringing into a group things which increase the assets of that group will certainly have the effect of increasing the group. Hence, $7+(+2)=7+2=9$.

This will probably give rise to a very natural question

which, if properly answered, disposes effectually of the whole difficulty concerning the Rule of Signs. The question will probably present itself in two ways; thus: "In what way does $(+2)$ differ from the plain figure 2?" and "How could things be brought into a group which would operate to *reduce* the assets of that group?" And the answer is as follows.

The first part of the question really amounts to a suggestion that the brackets and one of the plus signs of the given statement are redundant. But in ordinary business, payment may be made either (1) in actual things, or (2) by means of a credit note or other voucher. And, given good faith, either of these forms of payment is as good as the other. In mathematical work we assume perfect faith; and the credit note is often as convenient in such work as in ordinary business. The symbol "$+2$" means that we add *two actual things*; and the symbol "$+(+2)$" means that we add *a credit note (or voucher) for two things*. When we make the statement $7+(+2)=7+2$, we express our acceptance of the voucher as equivalent to an asset; and the same statement might be made in words thus: "A group of seven things, with the addition of a voucher for two further things, is *as good as* a group of seven things increased by two further actual things." Here, the sign "$=$" means *as good as*; and this is very frequently its meaning in practical work. There are occasions, however (as will be seen presently), when it means *the same thing as*, or *identical with*.

The second part of the question is now easily answered. Since we agree that a credit note or voucher is as good an asset as actual payment, we must also agree that a debit note or "bill" reduces assets just as effectually as an actual withdrawal of assets. Hence, things brought into a group *may* reduce the assets of that group, for *adding a bill* for a certain number of things is agreed to reduce the assets as effectually as actually *withdrawing those things*.

From this, the statement $7+(-2)=7-2=5$ will be

interpreted without difficulty, the plus denoting the *bringing in*, and the (-2) signifying *a bill for two things.*

Similarly, the statement $7-(+2)=7-2=5$ will present no difficulty, for it means simply that if a man possessing seven things issues a voucher for two of those things to be transferred to some one else, he has (in effect) reduced his assets to the extent of two things.

Lastly, the statement $7-(-2)$ means that from a man possessing seven things we take away a bill which demanded payment of two things. Now, if we are to take away a bill, it is clear that the bill (or one for a greater amount) must have been served previously; and hence it follows that, but for the bill, the man's assets would have been more than 7. On the basis of perfect faith, taking away (or discounting) a bill gives complete relief from the obligation to pay. Hence, in this case, the " $-(-2)$ " means that the assets are to be accounted 2 more than formerly; and it follows that $7-(-2)=7+2=9$.

Perhaps the matter will be seen more clearly from another point of view. Suppose that the assets recorded as 7 things consist of 12 actual things and a bill demanding payment of 5 things. The assets would be stated as $(12-5)$; and if now the bill be discounted by 2, the demand for payment will be reduced to 3, whence $12-3=9$. The important point lies in grasping the fact that a reduction of liabilities is *as good as* a corresponding increase in assets. It is, of course, not the same thing, for I can reduce a man's liability to me (by discounting a bill previously served by me upon him) without actually possessing a penny myself; whereas I cannot increase his assets (*i.e.* give him *things*) unless I actually have them to give. Moreover, in the latter case, things would be transferred from me to him; whereas, in the former case, he would merely be relieved from the obligation to transfer things from himself to me. But the effect on his assets, and also on mine, would be the same in both cases.

The foregoing argument may easily be extended to yield

a natural interpretation of such statements as $\{(+4) \times (+3)\}$ and $\{(-4) \times (-3)\}$. Some slight liberty is taken with the symbolism in making such statements; but as the realities remain simple and unchanged, and the liberties are taken solely for the purpose of convenience, there is no ground for objection to them. It only remains to secure a full and clear understanding of the statements and their meanings.

It will be well, first, to explain the liberty referred to above, which consists of introducing one pair of small brackets, thus making what is really an *enumeration of movements* (or *passages*) appear as either a voucher or a bill according to the circumstances. Without these brackets, however, the statements might be confusing, as the student will readily find if he will try to dispense with them—obtaining, for instance, the statement $\{(-4) \times -3\}$.

The statement $\{(+4) \times (+3)\}$ means simply that *on each of four occasions, a voucher for three things is added to a group.* Taking this sentence and comparing it with the given symbolical statement, it will be seen that—

(1) The *number of occasions* (or times) which the operation is performed is indicated by the figure 4.
(2) The *kind of operation* (whether *bringing into* the group or *taking from* it—in this case, the former) is indicated by the sign (in this case $+$) written with the figure 4.
(3) The *action of occasioning* (*i.e.* the *times* or *repetitions* of it) is indicated by the $\times$; and it will be clear that the statement would lose little or nothing in clearness if this sign were omitted.
(4) The *nature of the documents or things* brought into the group (in this case a *voucher*) is indicated by the sign (in this case $+$) written with the figure 3; and
(5) The *number of things* involved at each occasion or repetition is indicated by the figure 3.

Of course, it will be clear that the statement might equally well be interpreted as, *on each of three occasions, a voucher for four things is added to a group*; and the student

will have no difficulty in apportioning or detailing the symbols of the statement to the words and phrases of this interpretation on the lines indicated above.

Obviously, the addition of 4 vouchers, each for 3 things, will increase the assets of the group by 12 things; and therefore we state:

$$\{(+4)\times(+3)\}=(+12).$$

The statement $\{(+4)\times(-3)\}$ means that *on each of four occasions, a bill demanding three things is added to a group.* Comparing this sentence with the symbolical statement, as before, it will be seen that in this case the things brought into the group are *bills*, as indicated by the minus sign written with the figure 3.

Four bills, each demanding three things, are obviously equivalent to one bill demanding 12 things, the effect being to reduce the assets of the group by 12 things; and therefore we state:

$$\{(+4)\times(-3)\}=(-12).$$

The statement $\{(-4)\times(+3)\}$ means that *on each of four occasions, a voucher for three things is withdrawn from a group.* The minus sign written with the figure 4 indicates that the *kind of operation* here is a withdrawal from the group.

Clearly, to withdraw four vouchers, each for three things, is equivalent to withdrawing 12 things—or, alternatively, is equivalent to serving a bill for 12 things, and therefore we state:

$$\{(-4)\times(+3)\}=(-12).$$

Finally, the statement $\{(-4)\times(-3)\}$ means that *on each of four occasions, a bill demanding three things is withdrawn from a group, or discounted.*

Four bills, each demanding three things, are equivalent to one bill demanding 12 things; and if his single bill be withdrawn (*i.e.* totally discounted), the effect will be to *increase* the assets of the group by 12 things. Hence we state:

$$\{(-4)\times(-3)\}=(+12).$$

The student should consider this exposition of the Rule of Signs very carefully and thoroughly ; and satisfy himself entirely as to its truth. Apart from the advantage of a good understanding of the matter itself, he will obtain an insight into many other branches of the work which will be of the greatest assistance to him.

With the object of showing as clearly as possible the meaning of such statements as those considered above, and the relation of each symbol in the statement with the words and phrases of the interpretation, the last of the four

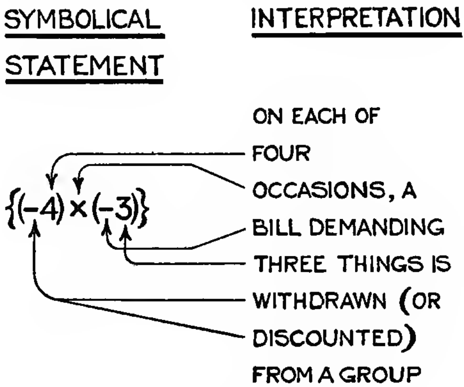

FIG. 1.

considered is illustrated diagrammatically in Fig. 1. The other three types should be treated similarly by the student for himself.

The statements would be in no way changed as regards either their nature or meaning by the use of letter-symbols in place of the figures ; for the letters would represent neither more nor less than do the figures, except that the latter are particular whereas the former are not.

Multiplication.—A good deal of the difficulty found in later work may be traced to vague or mistaken impressions regarding multiplication, which is too often treated as

though it were some peculiar process having no more than a remote connection with addition.

For example, it will be readily admitted that we cannot "multiply" an orange by 2. We can, provided we have it at our disposal, add a second orange to the first, making a group of two; but that is addition pure and simple. Yet many calculations are performed as though the even more impossible task of "multiplying" (say) 4 tons by 6 feet were rendered quite simple by shutting our eyes while some super-wizard transforms our real tons and real feet into that mysterious combination a "foot-ton," or the equally mysterious "ton-foot." It is comforting to reflect that, besides being unable, we have no need to do these impossible things.

Multiplication is not a separate and peculiar process in itself. It is merely a convenient method for performing addition where the conditions and circumstances are favourable—*i.e.* when the number of things forming a group is such that they are or may be arranged in a number of equal subgroups.

The statement $3 \times 5 = 15$ means, simply and solely, that if there be 3 groups, each group containing 5 similar things, the total assemblage comprises 15 things. This is illustrated in Fig. 2, each thing being represented by a small cube. From this it will be seen, also, that "3 groups of 5 things" is identically the same as "5 groups of 3 things"; for if the assemblage be viewed from either end it will appear as the former, and if viewed from either side it will appear as the latter. It is, of course, the same assemblage all the while, independently of the point from which it may be viewed.

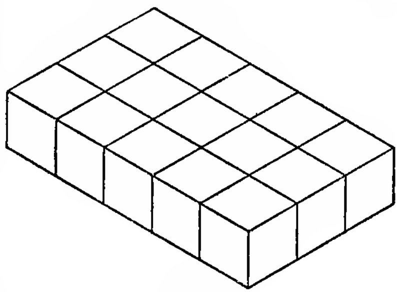

FIG. 2.

Similarly, $a \times b$ (or ab) means an assemblage of real things which are or may be arranged in a rows, each row consisting of b things. Again, $a \times b \times c \times d$ (or $abcd$) means an assemblage of real things capable of arrangement in a blocks, each block formed of b layers, each layer comprising c rows, and each row consisting of d things. And so on.

It is convenient to interpret 3×5 as " three multiplied by five " in ordinary speech and thought, and there is no harm in doing so provided the facts are properly understood and appreciated.

The student will do well to consider this use of the sign $\times$, and to compare it with that described above (see p. 34) in the discussion on the Rule of Signs. It will be found, doubtless, that " occasions " and " rows " are not very dissimilar in meaning.

The process of arithmetical multiplication is too simple and obvious to need elaboration here. For instance, in multiplying 468 by 37, we carry out the operation in two stages ; first finding the contents of 30 groups, and then of 7 groups, each group comprising 468 similar things, afterwards adding these two sets of groups together to count the number of things in the total assemblage.

Division.—The process of division is the reverse of multiplication. To divide 204 by 3 is merely to arrange 204 things in 3 *equal* groups, and then to count the number of things in each of these groups. At the same time, the result shows the number of groups into which the 204 things could be arranged, each group to contain 3 things, These two conclusions are, in fact, one and the same, as will be seen from Fig. 2.

No further discussion of division is necessary here, though the student will do well to consider algebraical division very carefully in the light of the above interpretation, and to convince himself of its general truth and applicability.

Brackets and Bracketing.—A mistake commonly made in studying " brackets " is the concentration of attention

upon the *removal* of brackets from given expressions; whereas for practical work it is by no means less important to acquire the ability to *introduce* brackets judiciously and effectively where such introduction is likely to be of assistance.

The principles involved are very simple, and a brief consideration will suffice to show them clearly.

The statement $3(2a+5b-7c)$ means that we have three "baskets," each basket containing two packets of a things each, five packets of b things each, and seven bills demanding delivery of c things each. If the contents of these three "baskets" were tipped out on to a table, and the similar packets and documents collected, there would be six packets of a things each, 15 packets of b things each, and 21 bills demanding delivery of c things each; and the assemblage would be recorded as $(6a+15b-21c)$.

Similarly, the statement $\{-5(3x-7y-4z)\}$ means that we are to take away from some group of things five basketfuls, each basketful comprising three packets of x things each, seven bills demanding delivery of y things each, and four bills demanding delivery of z things each. Clearly, the assets of the original group will be reduced by 15 packets of x things each; but the taking away (*i.e.* the discounting) of the bills will give an increase of assets for the group, amounting to $35y$ from the one set of discountings, and to $20z$ from the other. The consequent change in the assets of the original group would be recorded as $\{-15x+35y+20z\}$.

From this it is easy to see that an assemblage comprising $(9a+33b-12c)$ might be packed up into three "baskets," thus $3(3a+11b-4c)$. Also that an assemblage comprising $(abx+axy-xaz)$ might be arranged as $\{ax(b+y-z)\}$; or as $\{-ax(z-b-y)\}$, if such arrangement were likely to prove convenient.

CHAPTER III

FACTORS AND FACTORISATION

The Meaning of Factors.—Few things in mathematical work are more frequently useful than the ability to "factorise" a number or algebraical expression; and few things are easier or more straightforward in principle. Factorisation represents a mechanical operation; and a very simple one at that. Like all other mathematical processes, it is just a matter of groups and grouping.

Suppose we have 30 pebbles. They may be separated into 2 equal subgroups of 15 pebbles each; and these subgroups may each be rearranged in 3 rows, with 5 pebbles in each row. Now, 2, 3, and 5 are called the *Factors* of 30; and since no further subgrouping is possible without breaking up some of the pebbles, 2, 3, and 5 are said to be the *Prime Factors* of 30.

Again, if we had to transport 2310 horses, we might separate them into 6 batches, with 385 horses in each batch. We might spread the deliveries over 6 days, sending 385 horses per day. Further, 385 horses might be arranged in 5 groups of 77; and each of these groups might be arranged in 11 subgroups of 7 horses each. Thus, we might despatch on every day for 6 days, 5 trains, each comprising 11 trucks, with 7 horses in each truck. Again, 6 days may be regarded as consisting of 2 equal groups of 3 days each; and the deliveries might be spread over two weeks, working 3 days per week. The numbers thus obtained —2, 3, 5, 7, and 11—are the prime factors of 2310.

Factorisation, therefore, is simply the mechanical process of repeatedly arranging and rearranging groups of things into equal subgroups; and factors are the numbers of equal subgroups in which a given group of things may be arranged. When the things have been subgrouped to such an extent that no further reduction is possible without breaking up some of the things, the numbers of such subgroups are called prime factors.

We can tell by the "shape" (as it were) of a number whether the things it represents may be arranged in smaller subgroups having whole numbers of rows, and whole numbers of things in each row. These "tests for divisibility" are so well known, and are given in so many books, that there is no need to describe them here.

Algebraical Factors.—One very common and useful form

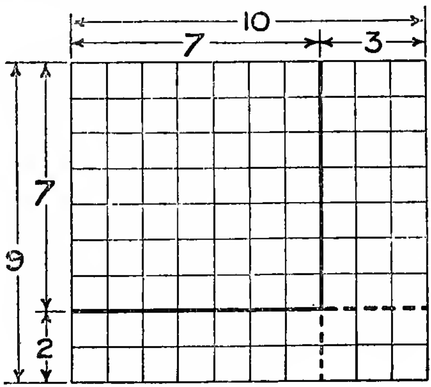

FIG. 3.

of factorisation is of the type, $x^2+7x+12=(x+3)(x+4)$; and this—though actually quite as simple and real as arithmetical factorisation—is often found difficult to understand, solely because the mechanical process of subgrouping is lost sight of.

Suppose we have a group consisting of 90 things. We may represent each thing by a small square, and we may arrange them in 10 rows of 9 things each, as in Fig. 3. We

might then notice that 49 of the things are standing in 7 rows, with 7 things in each row, as indicated by the heavy full lines in Fig. 3—what we might call a "square group," since the number of rows is equal to the number of things in each row—while the remaining 41 things form a sort of "letter L" group along two adjacent sides of the square group. These 41 things may, however, be regarded as forming three smaller groups, as separated by the heavy dotted lines; and we then have the 90 things arranged in four smaller groups indicated for clearness in Fig. 4, hatched, and lettered A, B, C, and D.

Now, the groups B and C are each 7 in depth; and

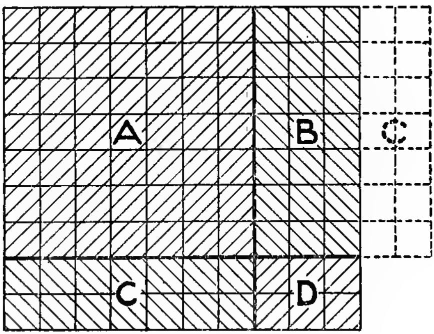

FIG. 4.

hence, if the group C were turned through a right angle and brought up to the right-hand side of the group B (as shown dotted in Fig. 4), it would form, with B, a single group of 5 (*i.e.* $3+2$) rows, each row consisting of 7 things. The group D comprises 3 rows of 2 things—*i.e.* 6 things.

Hence, we may say that the 90 things are arranged in groups thus: $49+7(3+2)+(3\times2)$; while the first grouping, 10×9, may obviously be written as $(7+3)(7+2)$. These are, of course, not only equal, but identical; and therefore, writing 7^2 in place of 49:

$$7^2+7(3+2)+(3\times2)=(7+3)(7+2).$$

Even in arithmetic such manipulation is often of very great service; but in algebra, where the number corresponding to the 7 in the foregoing example may be unknown, its utility is enormous.

Writing x in place of 7, the above expression becomes:

$$x^2 + x(3+2) + (3 \times 2) = (x+3)(x+2);$$

and since the left-hand side of this equation may be simplified, it follows that $(x+3)$ and $(x+2)$ are the factors of (x^2+5x+6)—which means neither more nor less than

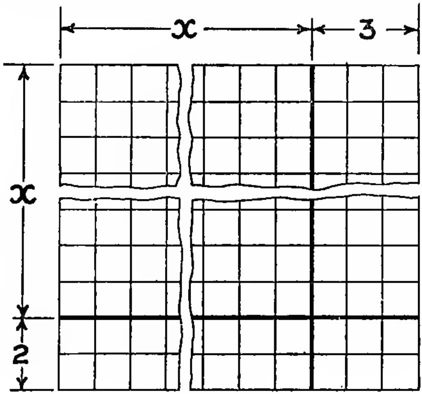

Fig. 5.

that a group comprising (x^2+5x+6) things may be arranged in $(x+3)$ rows, with $(x+2)$ things in each row. This is represented in Fig. 5, x being shown as indefinite since it is not known.

One important point should be noticed in passing. With the 90 things, arranged as in Fig. 3, we chose to take 7×7 for the group A; but we might equally well have taken 3×3, or 8×8, or any other square group not more than 9×9. With the algebraical expression, however, we have no such choice. The term common to both factors is x, whatever x may be.

So we see the relations between the component groups

and the whole assemblage. Now let us see how the factors of such an expression may be determined, taking for example $x^2+7x+12$.

First of all, it must be observed that the expression represents merely a number of real and similar things, arranged in three subgroups. One subgroup comprises x^2 things—*i.e.* some number of things capable of arrangement in x rows, with x things in each row. Another subgroup comprises $7x$ things—*i.e.* some number of things which may be arranged in 7 rows, with x things in each row. And the third subgroup comprises 12 things.

From this it follows that x^2 is of the same nature as $7x$; and that both are of the same nature as 12. This fact will be apparent from Fig. 4; and it cannot be either too clearly understood or too carefully borne in mind, for much trouble and confusion is caused by vagueness regarding such matters. Obviously, it would be absurd to attempt the "factorisation" of an assemblage comprising 16 fields, 28 fences, and 12 men, in view of the process which we have seen that factorisation really represents.

Let us take stock of the things with which we have to deal. Symbolising each thing by a small square, and remembering that x is some number which we are not required to particularise, the assemblage indicated by the expression $x^2+7x+12$ may be represented diagrammatically as in Fig. 6.

Now, from our experience of Figs. 3 and 4, it is clear that we have to divide our 7 rows of x into two groups which will lie along two adjacent sides of the x^2 group; and the corner space left (corresponding to D in Fig. 4) must exactly accommodate the remaining 12 things, suitably arranged.

Usually it will be found easiest to consider first what rearrangements may be made of the things forming the group which is independent of x—in this case 12—in order to see whether the number of rows and the number of things in each row, added together, will equal the number

of rows of x in the middle term—in this case 7. This, of course, amounts really to finding two factors of 12 which

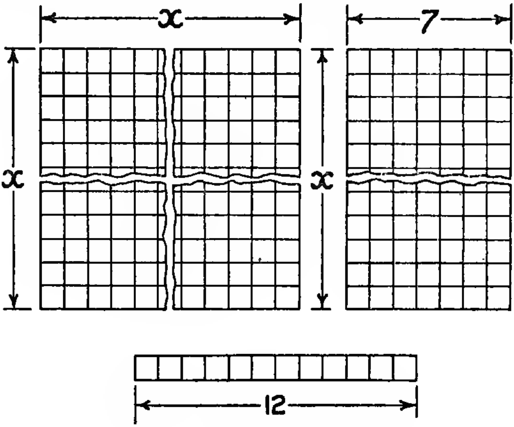

FIG. 6.

have 7 for their sum. The integral (*i.e.* whole number) factors of 12 are 1 and 12, 2 and 6, and 3 and 4; and of these only the last pair meet the requirements as regards sum. Separating the 7 rows of x into two groups, one having 3 rows of x and the other 4 rows of x, and placing them along adjacent edges of the x^2 group, the 12 things (arranged in 3 rows of 4) may be accommodated exactly in the right-hand bottom corner space, as indicated in Fig. 7.

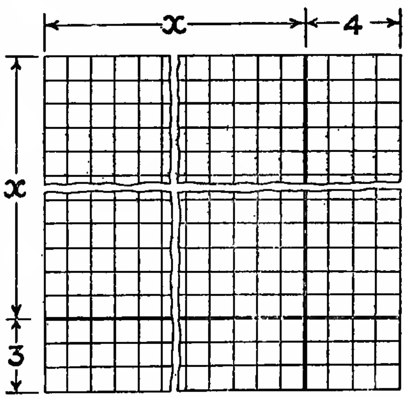

FIG. 7.

The assemblage has now been arranged in what may be termed a " rectangular group "—*i.e.* in some number of

equal rows—which is but another way of saying that the given expression has been "factorised." Clearly, from Fig. 7, the factors of $(x^2+7x+12)$ are $(x+3)$ and $(x+4)$.

Here an important point should be noticed. When the term which is independent of x has no pair of whole number factors which, added together, equal the coefficient of x in such an expression, it is too often assumed that the expression cannot be factorised; but this assumption is not necessarily true. So long as the expression represents an assemblage of things which *can* be arranged in a rectangular group, it may be factorised, *though not necessarily in whole numbers*. Sometimes fractional factors may be determined by the method described above; but, generally, other methods are more expeditious, as will be seen in a later chapter. The one condition which makes an expression of this type impossible to factorise is obtained when the expression represents an assemblage of things which *cannot* be arranged in a rectangular group—*i.e.* when there is no possible rearrangement of the things represented by the middle (or "x") term by which all the things of the last (or "constant") term can be accommodated. For instance, the expression $x^2+12x+37$ *cannot* be factorised; for the utmost accommodation which can be obtained by separating the 12 rows of x into two groups, and placing them along adjacent sides of the x^2 group, is sufficient for only 36 things (as indicated in Fig. 8), whereas we have 37 things in the assemblage. The importance of this point is so great that it cannot be too clearly understood—as will be seen from the treatment of quadratic equations, and "imaginary quantities," in Chapter VIII.

With the type of expression under discussion, we have so far considered only the form in which all the terms are positive. The three other possible variations are no more difficult to deal with than that examined and illustrated above; but as they combine with that to form a highly instructive and interesting aspect of factorisation, it will

be well to consider here a typical instance of each, on the lines already indicated.

To factorise the expression $x^2-9x+20$ we have to arrange in a rectangular group the things remaining after 9 rows of x things have been paid away from an assemblage previously consisting of two groups, one comprising x rows of x things, and the other comprising 20 things.

The delivery of the $9x$ things might be effected by paying away 9 of the rows of x from the x^2 group, leaving

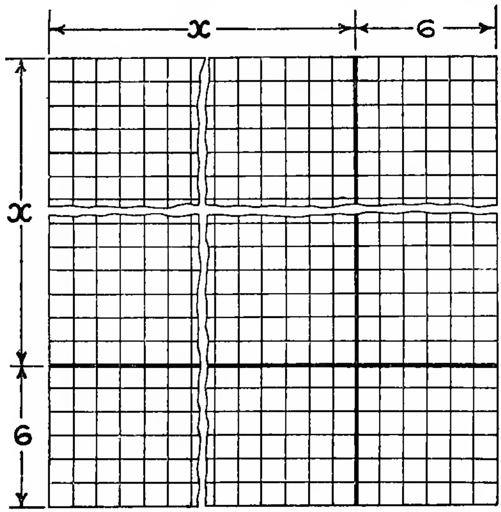

FIG. 8.

the shaded group of Fig. 9; but this would not allow us to add the 20 loose things in such a manner as to form a rectangular final group, since we do not know what relation x bears to 20. If, however, we took some suitable number (less than 9) of rows from the right-hand side of the x^2 group, as in Fig. 10, the 20 things might be used (as at ABCD in Fig. 10) to make up to the full contents (*i.e.* x things), a sufficient number of rows lying along the bottom edge of the x^2 group to complete the payment, while leaving

the remaining assemblage in a rectangular group. To take 6 rows of x from one side will not serve the purpose,

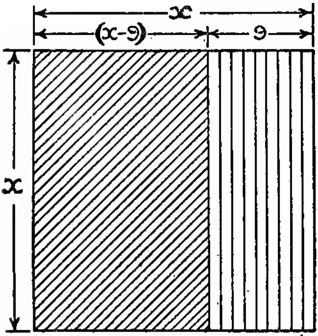

FIG. 9.

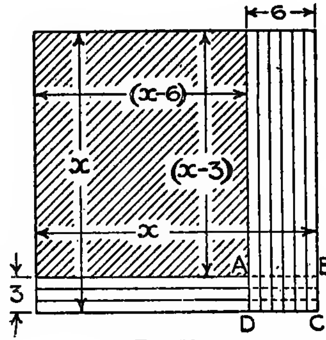

FIG. 10.

for we have only 3 more rows to pay away in full discharge of the debt; and this will use up only $6 \times 3 = 18$ of the 20 loose things, leaving 2 which cannot be accommodated in the final rectangular group. We require the pair of factors of 20 having 9 for their sum — viz. 4 and 5—and the factorisation is as indicated in Fig. 11, a group comprising $(x^2 - 9x + 20)$ things having been arranged in $(x - 4)$ rows, with $(x - 5)$ things in each row.

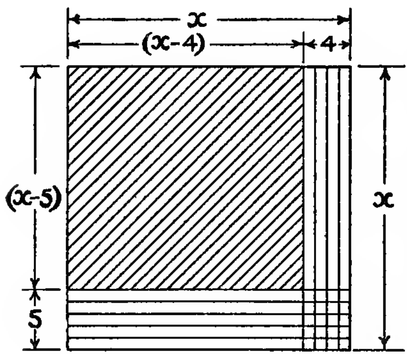

FIG. 11.

To factorise the expression $x^2 + 3x - 10$ we have to arrange in a rectangular group the things remaining after paying away 10 things from a stock comprising $(x + 3)$ rows of x things. Of course if we knew that x were greater

than $3\frac{1}{3}$ the 10 things might be paid away from the 3 rows of x things; but even then we could not arrange the remaining things in a rectangular group, except in the solitary and peculiar case of x being exactly equal to $3\frac{1}{3}$, when the x^2 group would be left without excrescences or reductions. Since we know nothing as to the particular numbers which x may represent in different circumstances, some other method must be found if the task is to be properly performed.

Now, if the $(x+3)$ rows of x things were arranged as in Fig. 12, some number of rows of x things (such as ABCD in Fig. 12) might be taken from the bottom edge of the x^2

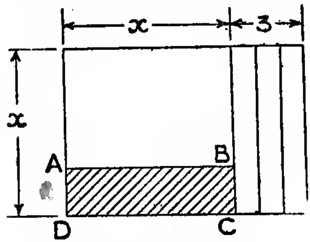

FIG. 12.

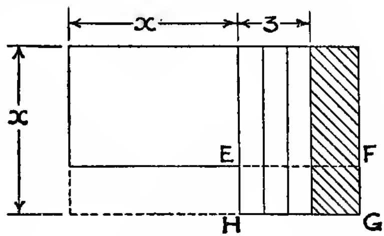

FIG. 13.

group and placed along the right-hand side of the 3 rows of x, as indicated in Fig. 13; and if the things standing in the group EFGH (Fig. 13) numbered exactly 10, they could be paid away in discharge of the debt, leaving a rectangular group. The factorisation would then be accomplished; and it only remains to see what number of rows must be taken from the bottom of the x^2 group and transferred to the right-hand side of the 3 rows of x things in order that the outstanding group EFGH shall comprise exactly 10 things. A very little consideration will suffice to show that we require those two factors of 10 which *differ* by 3—since the one is added to the x^2 group (towards the right in Fig. 13) and the other is subtracted (upwards) from it. Clearly, 2 rows of x must be taken from the bottom of the x^2 group and transferred to the right-hand side of the 3 rows of x, the

factors of $(x^2+3x-10)$ being $(x+5)$ and $(x-2)$, as shown in Fig. 14.

Lastly, to factorise the expression $x^2-4x-21$, we have to pay away 4 rows of x things, and also 21 further things, from a group comprising x rows of x things; and to arrange the remaining things in a rectangular group.

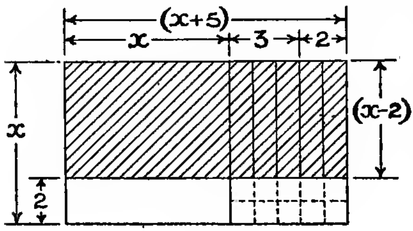

Fig. 14.

The debt of $4x$ may be discharged at once, by simply detaching 4 of the x rows of x things, as indicated in Fig. 15. Also, reasoning as before, the 21 things must be assembled at the right-hand bottom corner for detachment in discharge of the remaining debt so that the final assemblage may be left in a rectangular group.

If some further rows of x things were taken from the

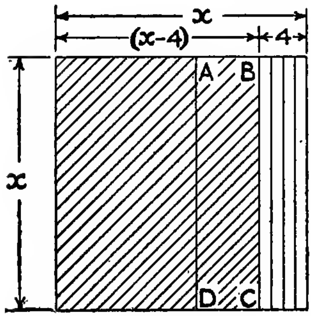

Fig. 15.

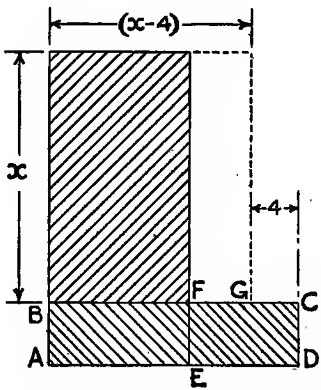

Fig. 16.

right-hand side of the $(x-4)$ rows—as indicated by the group lettered ABCD in Fig. 15—and transferred to the bottom edge of the group, as in Fig. 16, the outstanding

group CDEF might be detached, leaving the remainder in a rectangular group as required. It only remains, then, to ensure that the group CDEF shall comprise 21 things exactly, and the factorisation will have been accomplished.

Clearly, from Fig. 16, EF (which is equal to AB) and ED must be factors of 21—*i.e.* 1 and 21, or 3 and 7, if the factorisation is to be in whole numbers. Further, $ED = FG + 4 = AB + 4$, whence, $ED - EF = 4$. We require, then, the pair of factors of 21 which *differ* by 4—*i.e.* 3 and 7—and the factors of $(x^2 - 4x - 21)$ are $(x - 7)$ and $(x + 3)$, as shown in Fig. 17.

Much of the most useful factorisation in practical work may be effected by the collection and suitable arrangement of related terms, and the isolation of common factors, combined with judicious introduction of brackets, on the lines indicated in p. 39. Such work is often considered too simple to call for either serious thought or assiduous practice; but as a fact it is well worth all the attention which may be bestowed upon it, and will yield a handsome return for any trouble expended in bringing it under control and cultivation.

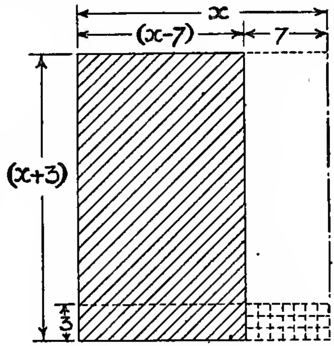

FIG. 17.

Standard Types of Factors.—There are a few standard types of factors which are of such wide and frequent applicability in practical calculations that a full comprehension of their significance is necessary where real facility and dexterity in mathematical work is desired. It will suffice if we consider four of these types here: (1) the perfect square; (2) the difference of two squares; (3) the difference of two cubes; and (4) the sum of two cubes. Others will be found treated in later chapters; and useful modi-

fications will doubtless suggest themselves readily as need arises, provided the standard types be properly understood.

The Perfect Square.—It is important to observe that the meaning of the word "square," in mathematical work, is not limited to the geometrical four-sided plane figure which has all its sides equal and all its angles right angles. By a "square" we mean *a group of similar things, capable of arrangement in some number of equal rows, so that the number of rows is equal to the number of things in each row.* Such an arrangement may conveniently be called a "square group"; and a little consideration will show that this

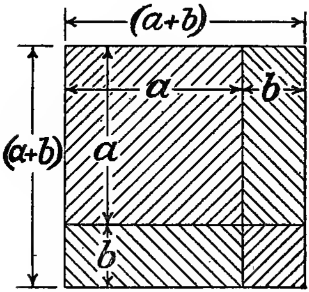

FIG. 18.

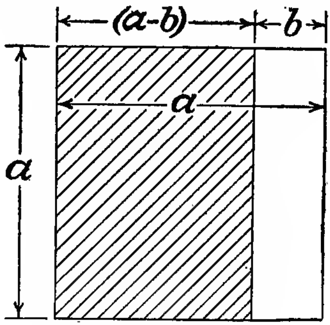

FIG. 19.

wider definition really includes the geometrical square, for the significance and import of such a figure is certainly not confined to its perimeter.

An assemblage consisting of $(a^2+2ab+b^2)$ similar things may be arranged in a square group, as shown in Fig. 18; and hence it is called a "perfect square," the number of rows, and also the number of things in each row, being $(a+b)$.

Similarly, an assemblage comprising $(a^2-2ab+b^2)$ things may be arranged in a square group. Applying the method for factorisation described in the preceding pages, we may take the a^2 subgroup (which, of course, comprises a rows, with a things in each row), and discharge half our debt by paying away b rows of a things from the right-hand side,

as indicated in Fig. 19. The b^2 subgroup may then be added at the bottom of the right-hand side, as in Fig. 20; and the remaining half of the debt may be discharged by paying away a rows of b things from the bottom of the group. This leaves the assemblage arranged in a square group, consisting of $(a-b)$ rows, with $(a-b)$ things in each row, as indicated in Fig. 21.

Useful modifications of these two cases will readily suggest themselves upon consideration, as also will other typical instances of the perfect square; and much useful information is to be acquired by examining and investi-

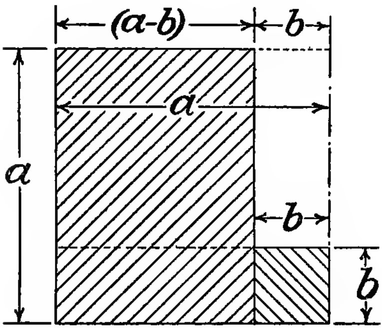

FIG. 20.

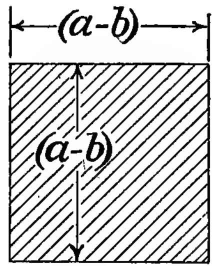

FIG. 21.

gating such groupings on common-sense lines. A method which the author can confidently recommend, as yielding most valuable assistance in the practical study of groupings and factorisation generally, is the use of simple and readily accessible appliances—such as a "Halma" board and "men." All the important cases illustrated here diagrammatically, besides many others, may be actually set up in a few moments by such means; and it will be found that impressions of the facts thus obtained are both real and permanent, as well as illuminating. One of the greatest advantages of this method lies in the fact that it focusses attention upon *groups of similar things* as the meaning of mathematical expressions, whereas they are too often

either not visualised at all, or thought of as relating to mere geometrical figures.

The Difference of Squares.—An assemblage consisting of (a^2-b^2) similar things—*i.e.* the things which remain after b rows of b things have been paid away from a group comprising a rows of a things—may be represented diagrammatically as in Fig. 22.

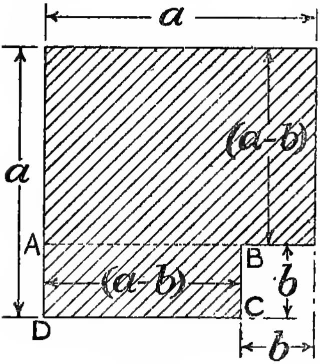

FIG. 22.

The subgroup ABCD may then be detached from the bottom edge, and replaced along the right-hand edge, as indicated in Fig. 23, leaving the assemblage "factorised" into $(a+b)$ rows, with $(a-b)$ things in each row.

It would be difficult to overestimate the importance and practical utility of this standard type of factorisation; and

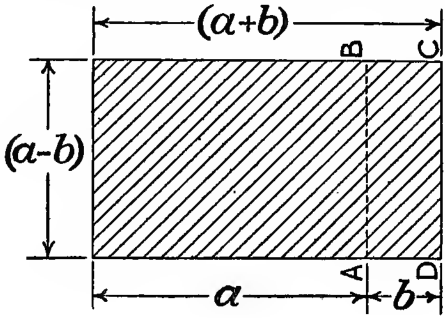

FIG. 23.

it is one of the easiest to demonstrate by means of the Halma board and men.

The Difference of Cubes.—Just as the word "square" means, for our purposes, a square group of similar things

rather than a mere geometrical figure, so " cube " may be taken to mean an assemblage of similar things capable of arrangement in a " cubical group "—*i.e.* a group comprising some number of layers, each layer containing some number of rows, and each row consisting of some number of things, the number of layers, the number of rows in each layer, and the number of things in each row, being all equal—rather than the geometrical solid which is bounded by six square faces.

If each thing in an assemblage comprising $(a^3 - b^3)$ things be represented (or " symbolised ") by a small cubical block, the assemblage might be arranged as in Fig. 24 ; and then, to " factorise " it, we have to rearrange the assemblage in some number of uniform layers at least. At most we might attempt a further rearrangement in which all the rows in each layer would be uniform (*i.e.* a " rectangular prismatic group "), as well as all the layers ; but this will be found impracticable in the case under consideration.

By means of a vertical division along the dotted line PPP in Fig. 24, the subgroup A may be separated from the remainder ; and this may be laid on one of its square faces, as at A in Fig. 25, forming a subgroup of $(a-b)$ layers, each layer comprising a rows, and each row containing a things. Further, by means of a horizontal division along the dotted line QQ in Fig. 24, the subgroups B and C may be separated. The subgroup B may be placed as in Fig. 25 by a simple movement of translation, without rotation ; while the subgroup C must be stood on end to bring it into position as in Fig. 25.

This completes the factorisation, for the whole assemblage has now been arranged in $(a-b)$ layers, each layer containing (a^2+ab+b^2) things—whence, we say that $(a^3-b^3)=(a-b)(a^2+ab+b^2)$.

If some particular and convenient relation were known to exist between a and b, the subgroup C (in Fig. 25) might be rearranged to lie evenly along the edge of the subgroup B ; and then the factorisation would have been extended

to give a rectangular prismatic group, with uniform rows

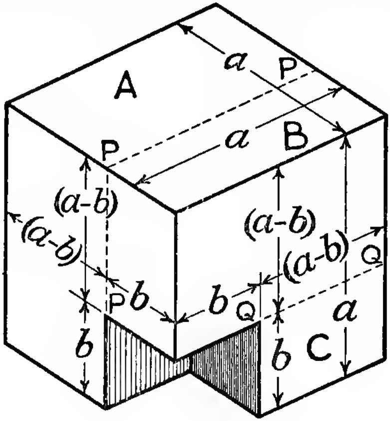

FIG. 24.

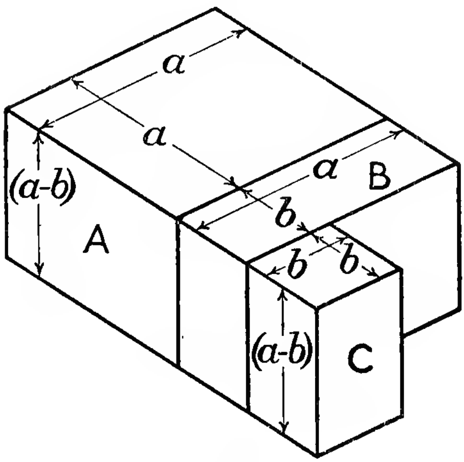

FIG. 25.

as well as uniform layers. As we have no knowledge of any such relation, however, the factorisation clearly cannot

be carried beyond the standard form as stated above and as illustrated in Fig. 25.

The Sum of Cubes.—On a basis similar to that adopted for the last preceding case, an assemblage comprising (a^3+b^3) things may be represented diagrammatically as in Fig. 26; and to "factorise" it, we have to rearrange the assemblage in some number of uniform layers—for in this case also it will be found impracticable to carry the process far enough to give uniform rows as well as uniform layers, in the absence of knowledge as to the existence of a particular relation between a and b.

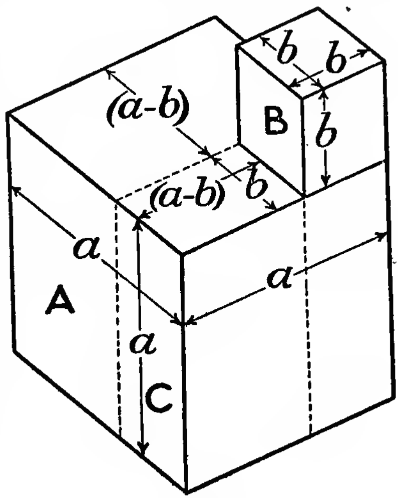

FIG. 26.

If the subgroup marked C in Fig. 26 be detached, the assemblage will be left in two portions, as indicated at (a) and (b) in Fig. 27; the a^3 subgroup having been reduced to the portion (Γ-shaped on plan) lettered A; the b^3 subgroup (marked B) remaining unchanged, both as regards shape and position; and the detached subgroup C—comprising a layers, with b rows in each layer, and $(a-b)$ things in each row—standing apart from the remainder.

Then if the detached subgroup C be laid on one of its largest—*i.e.* $a\times(a-b)$—faces, and rotated through a right angle in its horizontal plane, it will be found to fit exactly into the vacant space on top of the subgroup A, beside the subgroup B, as indicated in Fig. 28; and the assemblage is now of uniform height—*i.e.* $(a+b)$ layers—throughout.

A little consideration will show that the number of things in each layer is $\{(a^2-ab)+b^2\}$—for, ignoring for a moment the outstanding subgroup below B, the a rows of a things in the original a^3 group have been reduced by b rows of a things, leaving (a^2-ab) things, to which must be added the b rows of b things (*i.e.* b^2) forming the outstanding subgroup which culminates in B.

Hence the assemblage comprising (a^3+b^3) things has

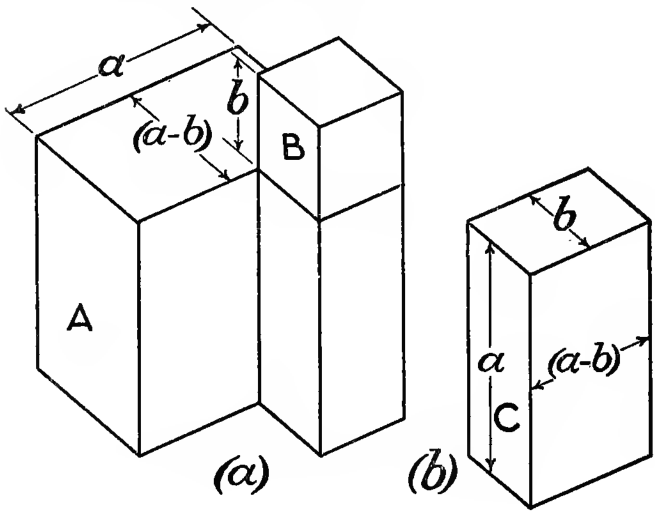

FIG. 27.

been "factorised" into $(a+b)$ layers, each consisting of (a^2-ab+b^2) things.

The "perfect cube," in its various practical forms, may be left to the interested student as a highly instructive example on the lines indicated above for the perfect square.

For the purpose of setting up these factorisations into "layers" and "rows" of "things," the Halma board may be employed with advantage, in conjunction with a supply of small cubes to fit on the squares of the board, each small cube symbolising a "thing" of the assemblage represented

by a given expression. By this means, the realities may be visualised; and they will be clearly and permanently impressed upon the mind in consequence.

Another method is to use cardboard models of the various subgroups, fitting them together to effect the desired factorisations. Such models have, however, one disadvantage as compared with the small cubes described above, in that they do not so clearly and constantly focus the attention upon the fact that they really represent *groups of similar things* in all cases.

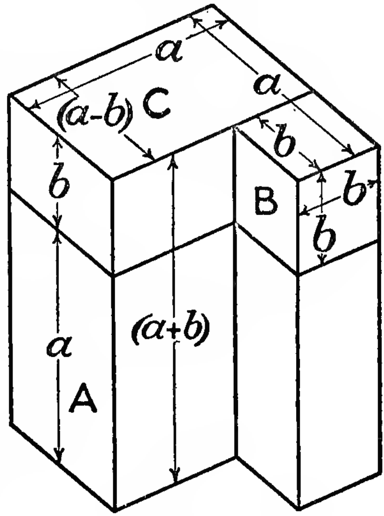

FIG. 28.

Highest Common Factor.—When dealing with two or more groups of similar things, it is often convenient to know what is the largest subgroup into which each of the given groups may be divided so that *all* the subgroups thus obtained shall comprise the same number of things.

As an arithmetical example, consider the case in which three groups are given, one comprising 8 things, one comprising 12 things, and the third, 20 things—all the "things" being, of course, similar. By the well-known method, the Highest Common Factor (generally referred to as the "H.C.F.," though sometimes spoken of as the "G.C.M.," or Greatest Common Measure) may be shown to be 4—which means simply that a subgroup comprising 4 things is the largest into which each of the given groups may be divided so that all subgroups shall be equal.

The operation might be recorded thus :

1 group of 8 things = 2 subgroups of 4 things ;

1 group of 12 things = 3 subgroups of 4 things ; and

1 group of 20 things = 5 subgroups of 4 things ;

and it will be clear that its effect is to indicate a new and more convenient "unit thing" in which the component assemblages may be expressed and dealt with.

The Highest Common Factor serves an important practical purpose in enabling us to deal with larger things in correspondingly smaller numbers than would otherwise be necessary ; and this is always the object of its determination.

Much of the work involved in dealing with expressions denoting groups of things in all branches of mathematics—algebraical, trigonometrical, etc.—may be simplified and facilitated by a judicious use of the Highest Common Factor ; and the interpretation of such operations, on the lines indicated above, may be left as an exercise for those sufficiently interested, who will find themselves well repaid for any trouble they may take regarding these matters.

Least Common Multiple.—When dealing with assemblages of things which are themselves portions of some larger thing, it is nearly always convenient—and often absolutely necessary—that the assemblages be expressed in terms of some smaller "unit thing," so that all may be sufficiently similar to admit of their being counted or compared under a single denomination.

As an arithmetical example, consider the case in which $\frac{3}{16}$ inch and $\frac{7}{20}$ inch are to be added, compared or dealt with in some other way together.

Here we have two assemblages ; one comprising 3 things, each of which is a sixteenth part of some unit thing (in this case a linear inch) ; and the other comprising 7 things, each of which is a twentieth part of the same (or an equal and similar) unit thing.

Now a sixteenth part of a thing is obviously unlike a twentieth part of the same thing ; and hence, the essential

condition of mathematics—viz. that the things concerned shall all be sufficiently similar to permit of their being classed under one denomination if they are either to be counted as in a single group, or if one subgroup is to be compared with another as regards their contents—is not satisfied as they stand.

If each sixteenth part be divided into 5 equal parts, each of these latter will be one-eightieth part of the original unit thing; and if each twentieth part be divided into 4 equal parts, each of these will be one-eightieth part of the original unit thing. Hence, the things comprised in both assemblages will then be sufficiently similar to lie within a single denomination; and they may, therefore, be treated by any appropriate mathematical process.

The Least Common Multiple (80, in this case) serves, then, an important practical purpose by indicating the smallest amount of sub-division which will bring dissimilar things within a common denomination for the purposes of counting or comparing them; and this is the sole object of its determination in practical work. The things, although dissimilar, must, of course, be all of the same kind —*i.e.* all lengths, or distances, all weights, all areas, and so on. No operation within human power will enable us to count a weight and a distance together, nor to compare a distance with an interval of time; but nobody—least of all the practical engineer—ever needs to do such weird and purposeless things.

In using the Least Common Multiple (or "L.C.M.," as it is generally termed) we have, perforce, to deal with larger numbers of "things," because the things themselves are reduced in size; but this is a small price to pay for the advantages obtained. The L.C.M. is, of course, quite as useful in dealing with algebraical and trigonometrical expressions as it is in simple arithmetical work; and the student should satisfy himself that he truly and fully appreciates both the reality and the significance of the operation when applied to expressions denoting groups

of things in those and all other branches of mathematics with which he may be concerned.

We are by no means finished with factors and factorisation. Though other chapters have headings which may seem, at first sight, to have no connection with factorisation, it will be found that many of them deal with operations which are in truth but little more than particular forms of factorisation—provided that the term be interpreted in that broad sense which has been shown above to reflect its true significance. Indeed, it is hoped that each succeeding page will make more clear the truth of the statement made in the opening chapter—that the whole of mathematics is concerned only with the counting of real and similar things in groups; and that all the operations of practical mathematics have for their object the arrangement and rearrangement of the things forming groups, to facilitate the work of counting or comparing them.

CHAPTER IV

INVOLUTION AND EVOLUTION

Involution.—As has been shown in the preceding pages, the symbol x^2 means simply a number of real and similar things, capable of arrangement in x rows, with x things in each row.

Much confusion and trouble arises through vagueness in thinking and speaking of this symbol as "x-squared," and arguing thence that *a square results from the multiplication of one length by an equal length at right angles.* We have seen that the ordinary human being cannot multiply even the most common thing of everyday life by 2; and it is therefore quite certain that he cannot multiply one length or distance by another. A length differs essentially from an area; and both of these are essentially different from a volume. When we speak of the area of a square having each of its sides 9 inches in length, we mean only the number of conventional "surface-units" bounded by the sides of the figure; and when we say that the area is $9^2=81$ sq. in., we mean neither more nor less than that there is an assemblage of real and similar things—each thing being an area of 1 sq. in.—arranged in 9 rows, with 9 things in each row.

If the difference between these two interpretations of the symbol x^2 were a difference of words only, there would be little need for the practical man to raise any serious objection or criticism; but there are extremely important issues involved in this matter, and it is therefore

necessary that the facts should be clearly seen and fully appreciated.

To assume that two lengths at right angles, multiplied together, give an area—*i.e.* the well-beloved "second dimension"—demands, as a logical consequence, that the multiplication of four lengths, all mutually perpendicular (in itself, a physical impossibility) produces the "fourth dimension," which the human intelligence—at least in its present stage of development—cannot conceive. Now, in a world so full of strange things as this, where every day the impossible is being achieved, and the inconceivable being realised, to say that there is no such thing as a fourth dimension to our space would be unwarrantable; but at the same time it is difficult to see why so simple and natural a process as Involution should be regarded as vague and mysterious on that account.

In practical engineering a conclusion arrived at by means of logical argument is tested as soon as possible by comparison with ascertained fact; and if the conclusion be found at variance with fact, the argument is immediately re-examined, with the object of locating either an error in the reasoning or a fallacy in the basis of assumption—and there must be at least one of these if the conclusion prove to be not in accordance with fact. If this method be applied constantly in the study of mathematics, such difficulties as arise from the confusion of mere "groups of things" with theories of "n-dimensional space" will be entirely avoided.

The symbol x NEVER represents a length in practical mathematics. It represents always *a number of real things*. When we use x (or any other symbol representing a number of things) to specify a measurement of length, it denotes merely the number of inches, feet, miles or other conventional units of length which must be placed contiguously to reach from one extremity to the other of the length or distance under consideration.

Our conceptions—or perceptions—of length, area and

volume are discussed in a later chapter, which it is hoped the student will read carefully; but in the meantime it cannot be too clearly understood that areas and volumes are *not* the products of either lines or lengths—at least, if the word "product" be taken (as it is clearly intended in such cases) to mean the result of multiplication.

Interpreting x^2 as the number of similar things in an assemblage which is capable of arrangement in x rows with x things in each row, the whole process of Involution is simple; and all deductions obtained logically from it on this basis are easily intelligible and readily verifiable.

Consider this case: A man is to visit two places, at each of which there are two pens, each containing two pigs; and to each pig he is to give two potatoes.

The particulars may be set out more clearly thus:

2 places are to be visited, there being
2 pens at each place, each pen containing
2 pigs, each of which is to be supplied with
2 potatoes.

We may then state that the number of operations to be performed is $2 \times 2 \times 2 \times 2$, which we may conveniently write as 2^4; but there is nothing "fourth-dimensional" in that! It is simply the *number of potatoes* which the man will require if he is to perform the specified task.

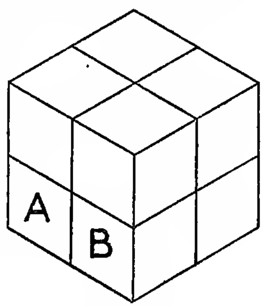

FIG. 29.

The potatoes might be arranged methodically thus, representing each potato by a small cube: Place the ration per pig—*i.e.* 2 potatoes—in a row, as A and B in Fig. 29; and since there are two pigs in each pen, 2 such rows will be required for each pen visited. As there are two pens at each place, 2 such layers will be necessary for a visit to one place; and this gives a group (or "block") of 8 potatoes, arranged—or capable of

arrangement—in 2 layers, each layer formed of 2 rows, and each row comprising 2 potatoes, as shown in Fig. 29. Since there are two places to be visited, 2 such "blocks" will be required, as indicated in Fig. 30, which represents the total provision necessary—*i.e.* 16 potatoes in all.

We use the "index" 4 in the above example because the 2 occurs as *a factor of repetition* 4 times. If a general expression were desired, applicable to any number on occasion, instead of to the particular number 2 only, the total number of potatoes necessary might be written as x^4; but, of course, there must then be x places, x pens

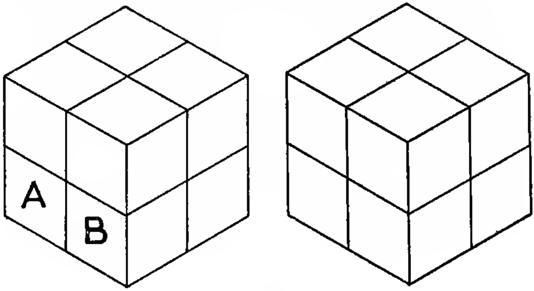

FIG. 30.

at each place, x pigs in each pen, and x potatoes to form the ration per pig.

The principle which underlies Involution is this: We start always with *one unit thing*, whatever that unit thing may be. We cannot increase it; nor can we either annihilate it or change its nature. We can only add to it other similar things from a store at our disposal, as explained in that portion of Chapter II. which deals with Addition.

Let us adopt the conventional statement that when, say, 9 *other* things have been added to (*i.e.* grouped with) the first, the unit thing has been "repeated to 10 times." This is strictly true, of course, for the group then comprises 10 times as many things as it did when it consisted of the single unit thing.

Now, it has been shown above that x^4 denotes merely a number of things which may be so arranged that x occurs 4 times as a subgrouping factor—or factor of repetition—the unit thing being repeated to x times at each occurrence of x. The original unit thing is to be repeated (by simple addition from a store, of course) to x times to form a *row*; each thing in that row repeated to x times to form a *layer*; each thing in the layer repeated to x times to form a *block*; and each thing in the block repeated to x times to form a *row of x blocks.*

For an assemblage of things denoted by x^7, the argument would be exactly as above so far as to include x^4, after which it would proceed on the same lines as follows: each thing in the row of x blocks of things repeated to x times to form a *layer of blocks* (represented by x^5); each thing in this layer of blocks of things repeated to x times to form a *block of blocks* (represented by x^6); and each thing in this block of blocks repeated to x times to form a *row of x blocks of blocks of things*, represented by x^7.

For any other positive integral index greater than unity, the interpretation follows precisely the same line of argument as that employed above; and no further explanation for such cases will be necessary. With a good supply of small cubes available, letting each cube represent a "thing," any ordinary case of Involution for an integral index of reasonable magnitude may be actually set up in a few minutes; and in spite of its simplicity, this is well worth the while of even advanced students who have not thoroughly satisfied themselves that Involution is merely another way of counting real things in groups, like the rest of mathematical processes.

The symbols x^1 and x^0 usually present some difficulty when approached from the orthodox standpoint; but if they be regarded as representing assemblages of real things obtained by repetition of the unit thing, their significance is perfectly obvious at once.

We have seen that in forming an assemblage which

comprises x^7 things, x occurred as a *factor of repetition* 7 times; in forming an assemblage to comprise x^4 things, x occurred as a factor of repetition 4 times; and so on. Clearly, then, in forming an assemblage to comprise x^1 things, x must occur *once only*—so that the original unit thing is repeated to x times *once*, forming a single row of x things; whence, $x^1 = x$ things.

Finally, in forming an assemblage to comprise x^0 things, x must occur as a factor of repetition *no times*—that is, *not at all*—so that the original unit thing remains (just the one thing itself) without repetition; whence, $x^0 = 1$ thing. The elaboration of this argument, and its application to groups of real things with specific values allocated to x, is commended to the careful attention of all who would acquire a sound working knowledge of mathematics for the purposes of practical engineering.

Manipulation of Positive Integral Indices.—There are three, so-called, "Laws of Indices"; and these are too often accepted by students without a proper comprehension of their significance and reality.

The first of these "Laws" may be expressed symbolically thus, $x^p \times x^q = x^{(p+q)}$; and also $x^p \div x^q = x^{(p-q)}$.

The second "Law" may be similarly stated thus, $(x^p)^q = x^{(p \times q)} = x^{pq}$; and a little consideration will show that this is but an elaboration of the first "Law."

The third "Law" may be expressed thus, $(x \times y \times z)^p = x^p \times y^p \times z^p$; and the operation of this, on orthodox lines, needs no explanation here.

Now, the readiness with which these "Laws" are accepted is due (in some cases, at least) to the fact that they provide an easily learned trick in mathematical jugglery. They are so simple to use on the lines of an automatic machine—needing only the pressing of certain buttons and the pulling of certain levers in a prescribed sequence—that no attempt is made to understand their real meaning. The author has not even the slightest desire to press mere microscopical precision for its own sake—indeed, it is

hoped that the treatment so far has shown with sufficient clearness that the object here is strictly utilitarian throughout; and the sole reason for drawing attention to this matter lies in the fact that any lack of proper understanding at this stage is almost certain to cause trouble and difficulty later, especially to those who have studied Pure Mathematics deeply on orthodox lines, and would apply mathematics in practical and commercial engineering with true knowledge and confidence.

The difficulty in question is likely to arise when an attempt is first made to apply the above-described method of interpreting the symbolical statements of Involution to such cases as $x^4 \times x^3 = x^{4+3} = x^7$; and it will probably take the form of a question such as this: " x^4 denotes an assemblage of things; so does x^3; and so does x^7. How can one assemblage of things be multiplied by another assemblage of the same kind of things, to produce a third assemblage? For example, if $x=2$ apples, then $x^4=$ 16 apples; $x^3=8$ apples; and $x^7=128$ apples. How is it that, while we obviously cannot multiply even one apple by 2—much less 16 apples by 8 apples—the given statement seems to indicate that we can? And further, how is it that, if we ignore the physical impossibility of multiplying apples by apples, and say simply 16 apples $\times$ 8 (*i.e.* a group of 16 apples *repeated to* 8 *times by addition from an available store*), the result of the given statement is in agreement with demonstrable fact?"

It is this very question, presenting itself in various forms, which has caused so much of the trouble arising from the usual presentment of "pure" or "abstract" mathematics—or, rather, it is the lack of a convincing answer to the question, supported by demonstration in the light of physical fact as experienced in everyday life. The author hopes that the following explanation may be found at least sufficiently suggestive of the true facts to enable the average student to reason the matter out for himself, and to demonstrate the facts, in any case likely to arise in

the ordinary course of practical engineering work. Such matters need very careful handling to avoid both the omission or distortion of essential facts on the one hand, and the drawing of unwarranted and unjustifiable conclusions on the other hand. Given the requisite care, however, and a sincere desire to see things as they are, there is nothing difficult in this matter. It is hoped that the following explanation will be read as merely an attempt to suggest the lines on which the student may reason the question out with himself; and not in any way as either claiming or pretending to be a complete and final exposition of the whole matter. In all the affairs of life, a fact is, and must be, interpreted by the individual as seen from his own particular standpoint, and in the light of its effects or influences upon him. For instance, the velocity of a motor-bus moving along a stretch of road could be measured with considerable accuracy, and the result demonstrated by reference to the time occupied in covering a certain distance; but if there be two pedestrians, each desiring to cross the road in front of the bus, the speed of the vehicle will be gauged by each according to his own powers of locomotion. Both pedestrians may be highly skilled in the observation of velocities, and both might agree that the bus is moving at the rate of, say, about 8 miles per hour; but if one be a tall and athletic young man, the crossing may be for him an easy and perfectly safe matter, while if the other be very short and elderly, the crossing may be for him obviously dangerous or impossible. Real education cannot be grafted on to a man from without by merely pumping into him formulated statements of fact—no matter how true, precise and comprehensive those statements may be. He must educate himself—which is but another way of saying that he must develop his own individuality—by perceiving the facts with his own senses, and interpreting them in the light of their effects and influences upon his own personality.

The answer to the question raised above is this: The

statement $x^4 \times x^3$ means that the original unit thing is to be repeated to x^4 times to form an assemblage ; and then, each thing in that assemblage is to be repeated to x^3 times, giving an assemblage which may be shown to comprise exactly the same number of things as if the original unit thing had been repeated to x^7 times straight away. Indeed, if the matter be considered with a little care, it will be seen that the second manner of repetition (*i.e.* to x^7 times without a stop) is precisely the same as the first if the temporary halt at the x^4 stage of repetition be disregarded.

In exactly the same way, $5 \times 3 \times 7$ does not mean 5 things multiplied by 3 other things, and the product multiplied by 7 yet other things. It means simply that the original unit thing is to be repeated to 5 times to form an assemblage of 5 things ; then each of those 5 things is to be repeated to 3 times, giving an assemblage of 15 things ; and lastly, each of those things repeated to 7 times, forming an assemblage of 105 things.

A perfectly clear comprehension of these points will be found of incalculable value ; and the student is strongly advised to read through again at this stage, and on these lines, the treatment of addition, multiplication, etc., in the preceding pages. In case it may be thought that over much importance is attached here to the arrangement of things in *rows, layers, blocks,* and so on, it may be well to point out that while such arrangement is not strictly necessary, it will be found preferable to any other course both in thinking out the various processes by means of mental visualisation, and also in actually setting up the assemblages represented by typical expressions, using small cubes or other pieces as symbols.

Now, it may seem that the foregoing answer to the question raised in anticipation is so simple and obvious that there can be little ground for anxiety concerning its full comprehension, and little reason to fear that it may lead to serious pitfalls if not properly understood ; but such is the fact, nevertheless, as will be seen on further consideration.

On the basis that x^4 means that an original unit thing has been—or is to be—repeated to some number of times, it is possible to argue that x^4 signifies a number of *repetitions*, and not that number of *things* ; and to a certain extent such argument is true, for the *action of repeating* is, of course, something quite distinct and different from the *thing repeated.* The argument is, however, pressed further. It is contended that, since the operation of repeating may be performed upon any kind of things, *the things should be abstracted,* so that the expressions may be quite general—*i.e.* entirely without reference to any special thing or kind of thing in particular ; and in this lies the danger—and the fallacy—which must be guarded against with scrupulous care.

It is both dangerous and fallacious to assume that "numbers" may be dealt with apart from the assemblages of real things to which they relate ; for numbers can have no significance of their own. Number is merely the result of counting things ; and the figures and other characters which we use in mathematical work are no more than labels which men have agreed shall represent to them the results of countings. The action of counting can no more be divorced from the things counted than can the action of eating from the things eaten, or the sensation of sight from the things seen. We may employ symbols (which may be either mental impressions or physical things subject to the same natural laws as are the things symbolised) to assist us in applying the results of actual countings to properly permissible calculations regarding things not under our immediate control ; but all our countings must relate to things either real or realisable, standing for themselves or as symbols representing other things.

We could, of course, argue that our interpretation covers—and should therefore prevent—the fallacious assumption that x^4 (for example) indicates a mere number of repetitions, for even an action of repeating is surely a thing ; and whether we count the actions or the things assembled in

consequence, the result will be the same. There is, however, no need to press the point, so long as the student is made fully aware of the danger by exposing the fallacy; and it only remains for him to think the matter out for himself completely, applying the inferences which he draws to explain cases arising in actual life, treating the cases of things which occur in sequence (one replacing another) as well as those in which things are brought together in an assemblage or group at one time.

Evolution.—The process known as Evolution—or root-extraction—is really simple factorisation, with the added condition that all the factors (or "roots") shall be equal.

Evolution is also the reverse of Involution; for in the latter we count the number of things in an assemblage formed by repeating the original unit thing to some specified number of times, while in the former we rearrange an assemblage of things into component subgroups to a specified degree or order of subgrouping.

Thus, the *second* (or *square*) root requires *two* degrees of subgrouping—in *rows* of *things*; the *third* (or *cube*) root requires *three* degrees of subgrouping—in *layers* of *rows* of *things*; and so on.

To find the square root of a number means simply that, given that number of similar things, we have to arrange them in a "square group"—*i.e.* in a number of equal rows, so that the number of rows shall be exactly equal to the number of things in each row. For instance, an assemblage comprising 100 men might be arranged in 2 rows of 50, in 4 rows of 25, in 5 rows of 20, or in 10 rows of 10; and the last of these, being a square group, gives 10 as the square root of 100.

Here, again, it is most important that the difference between a "square"—*i.e.* the geometrical figure known as a square—and a "square group" should be thoroughly understood; as also the fact that the latter term really includes the former, as is shown in p. 63. If 9 similar

sticks of pencil be arranged in 3 rows of 3, the sticks being in contact end-to-end and side-by-side, as in Fig. 31, they form a *square group* ; but certainly the assemblage is not a *square*. A square group becomes a square in the particular case when each thing in the group is itself a square. Sometimes, of course, we allow each thing to be *represented diagrammatically* by a square, for the purpose of mere illustration ; but the real nature of the things so represented should be constantly borne in mind, for to *represent* a thing by a symbol is not equivalent to transforming the symbol into the thing which it represents—and what a world of trouble would be prevented, in all departments of life, if human beings appreciated this simple fact !

The cube root of a number is found by effecting the arrangement of that number of things in layers of rows of

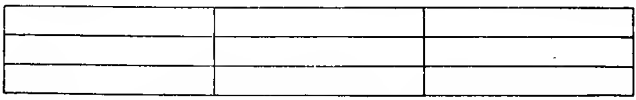

FIG. 31.

things, so that there are just as many layers as there are rows, and just as many rows as things in each row. The difference between a " cubic group " and a " cube " follows naturally on the lines indicated above with regard to the square group and the square ; and the student will do well to satisfy himself completely that he understands these matters thoroughly in the light of their applicability to groups of real things.

Fourth and higher roots are just as simple, in principle, as the lower roots. For instance, the fifth root of a number is determined by effecting the arrangement of that number of things in *rows of " blocks " of things*, as indicated in Fig. 32 (which represents an assemblage comprising 3^5 things, each thing being a small cube), with the number of rows of blocks equal to the number of blocks in each row ; and equal, also, to the number of layers in each block, the

number of rows in each layer, and the number of things in each row.

One fact, which follows clearly from the foregoing discussion, should be carefully noticed, proved (by testing its applicability to groups of real things), and fully appreciated as a basis for practical calculations.

The n-th root of a given number is of the same nature as the given number, and so, also, is the n-th power of that number;

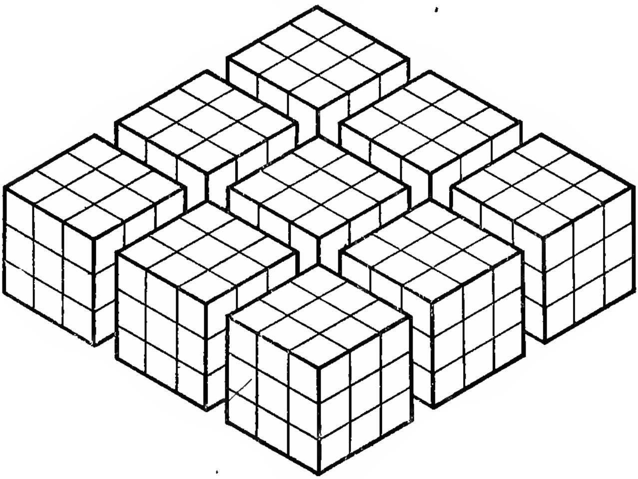

Fig. 32.

for all three relate to groups of the same kind of things. The square root of 100, if the 100 things be *men*, is 10 *men*; the square root of 9 *pencils*, in the case illustrated (*see* Fig. 31) above, was found to be 3 *pencils*; and so on.

The square root of an *area* is an *area*. For an area comprising, say, 16 square inches is but an assemblage of 16 things, each thing being an area of one square inch; and 16 things can be arranged to form a square group in one way only—*i.e.* so that there are 4 rows, with 4 things in each row. Hence, the square root of 16 square inches is 4 square inches.

Similar arguments may be applied to cube roots, and to all other degrees of Evolution, as well as to all degrees of Involution; and the student should follow out these arguments in their application to all the important cases arising, or likely to arise, in his practical work.

Combination of Involution and Evolution.—It should be noticed that Involution and Evolution are not limited to integral powers and roots. They may be combined, giving fractional indices, without any loss of simplicity as regards principle, and without in any way altering or affecting the nature of the things forming the groups or assemblages concerned. For example, the meaning of $\sqrt{4^3}$ is perfectly

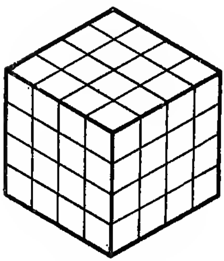

FIG. 33.

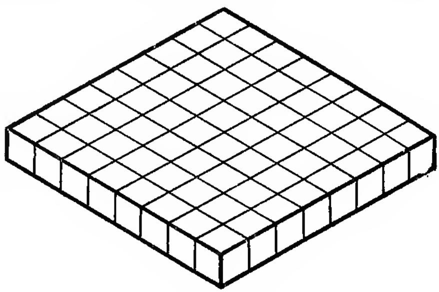

FIG. 34.

plain and simple. We should, for convenience, represent the number as $(4^{\frac{3}{2}})$, for reasons which are explained in Chapter V.; but the operation involved is simply to repeat some original unit thing (by addition from an available store of those things) to 4 times to form a row, repeat each thing in the row to 4 times to form a layer, and repeat each thing in the layer to 4 times to form a block, as indicated in Fig. 33. Then, laying this assemblage of things out flat on a table, we have to rearrange them in a square group—which may be effected as in Fig. 34. Since $4^3 = 4 \times 4 \times 4 = 64$, and $8^2 = 8 \times 8 = 64$, it follows that $(4^{\frac{3}{2}}) = 8$.

In the same way, there would be no difficulty (as regards principle) in finding $\sqrt[3]{3^5}$ which we should write as $(3^{\frac{5}{3}})$.

For we should merely have to take the total assemblage of things indicated in Fig. 32, and rearrange them to form a cubic group.

Fractional indices require careful handling, as is shown in Chapter V., to avoid the danger of vagueness or misunderstanding. So long as it is known and realised that $(4^{\frac{3}{2}})$ means what it has been shown to mean above, there is no harm in writing it as $(4^{1 \cdot 5})$ for convenience in treatment by logarithms or otherwise; but trouble is likely to arise in many ways if such methods of writing be employed without a clear comprehension as to the underlying realities.

The Physical Process of Evolution.—Perhaps in no other branch of mathematical work is logical reasoning and calculation more truly valuable and effective than in that branch which is based upon Evolution.

In all the processes discussed in the preceding pages, it has been shown that, if the numbers be particularised and the things represented by convenient symbols, the operations may be readily performed by actual arrangement and rearrangement, and the things in the resulting assemblages counted with very little trouble.

To determine by similar means the square root of a number—say 5, or 72, or 49106—is, however, a matter of considerably greater difficulty. Even with numbers which are the squares of integers, if they be large numbers, the determination of their square roots by physical means alone might prove somewhat troublesome; for, if logical argument be barred effectually, the process would have to be based more or less upon trial and error.

By the same methods, it will be clear that to determine the cube root of, say, 421·875 would be likely to prove a tedious and lengthy task, in spite of the fact that the result, when found, is a quite simple number; while it is probable that the determination, solely by means of actual arrangement and rearrangement of things without logical argument, of, say, $\sqrt[5]{5127}$ might safely be set by the editor of some popular newspaper as a "puzzle" for the amusement

of his readers, with the offer of a large prize for every solution obtained within a week.

In practical work, it is seldom that roots of higher degree than the third (or cube) are required—as is only to be expected, from the nature of the things and operations concerned—but some of those which have more or less frequently to be determined would be at least very troublesome to find if argument and calculation were not available.

At the same time, however, it is of the most supreme importance that it be clearly understood, by every one who would employ such arguments and calculations with true discernment and confidence, that the argument and calculation does—and obviously must—in all cases follow precisely the lines of actual arrangements and rearrangements of real things.

Perhaps this may be seen clearly from an examination of the ordinary arithmetical method for square root.

Let us consider the determination of $\sqrt{612 \cdot 5625}$ as an example.

Marking off the digits in groups of two, starting from the decimal point and proceeding in both directions, the number is arranged for logical treatment thus : $6\overline{12}\cdot\overline{56}\,\overline{25}$; and the calculation is usually set out as follows :

$$
\begin{array}{rl}
2) & 6\overline{12}\cdot\overline{56}\,\overline{25}(24\cdot75 \\
 & 4 \\
 & \overline{} \\
44) & 212 \\
 & 176 \\
 & \overline{} \\
487) & \cdot3656 \\
 & 3409 \\
 & \overline{} \\
4945) & \cdot24725 \\
 & 24725 \\
 & \overline{} \\
 & \cdot\cdot\cdot\cdot\cdot
\end{array}
$$

This is, however, an abbreviation of the actual process, many ciphers being omitted—a perfectly permissible and proper means for saving time and trouble, *provided the*

realities be fully understood; but a dangerous practice if the omission be made in ignorance of the things omitted.

The calculation might be set out thus :

$$
\begin{array}{rrrl}
20{\cdot}00\ ) & 6\overline{12}{\cdot}\overline{56}\,\overline{25} & (\ 20{\cdot}00 & \\
 & 400{\cdot}0000 & & 20{\cdot}00+4{\cdot}00 \\
\cline{2-2}
 & & & =24{\cdot}00 \\
20{\cdot}00\times 2=40{\cdot}00\ ) & 212{\cdot}5625 & (\ \ 4{\cdot}00 & \\
40{\cdot}00+4{\cdot}00=44{\cdot}00 & 176{\cdot}0000 & & 24{\cdot}00+0{\cdot}70 \\
\cline{2-2}
24{\cdot}00\times 2=48{\cdot}00\ ) & 36{\cdot}5625 & (\ \ 0{\cdot}70 & =24{\cdot}70 \\
48{\cdot}00+0{\cdot}7=48{\cdot}70 & 34{\cdot}0900 & & \\
\cline{2-2}
24{\cdot}70\times 2=49{\cdot}40 & 2{\cdot}4725 & 0{\cdot}05 & \left\{\begin{array}{l}24{\cdot}70+0{\cdot}05\\ =24{\cdot}75=\sqrt{612{\cdot}5625}\end{array}\right. \\
49{\cdot}40+0{\cdot}05=49{\cdot}45 & 2{\cdot}4725 & & \\
\cline{2-2}
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & &
\end{array}
$$

and the argument on which this is based will, it is hoped, be clear from the following description, with reference to the fully-set-out calculation and to Figs. 35, 36, 37, 38 and 39.

The argument, as will be seen, is based throughout upon the fact that $(a+b)^2 = a^2 + 2ab + b^2$.

Seeking first to determine the number which corresponds to a^2 in the above expression, we notice that 400·0000 is the largest number of things (being the square of an integer which is itself an integral multiple of 10) which is comprised as a square group in the assemblage represented by the given number. This number (400·0000) is subtracted, as shown in the calculation, and its square root (*i.e.* 20·00) recorded.

Withdrawing the 400 things from the total assemblage (which latter, for clearness in illustration, is indicated as a square group, as though its square root were already known), as shown in Fig. 35, leaves the 212·5625 remaining things in an L-shaped group. This group consists clearly of three portions, as indicated by the heavy full lines and distinctive hatching in Fig. 35; two of these subgroups, comprising each 20 rows, with some number—say p—of

things in each row, while the third comprises p rows, with p things in each row. We have to determine the number p.

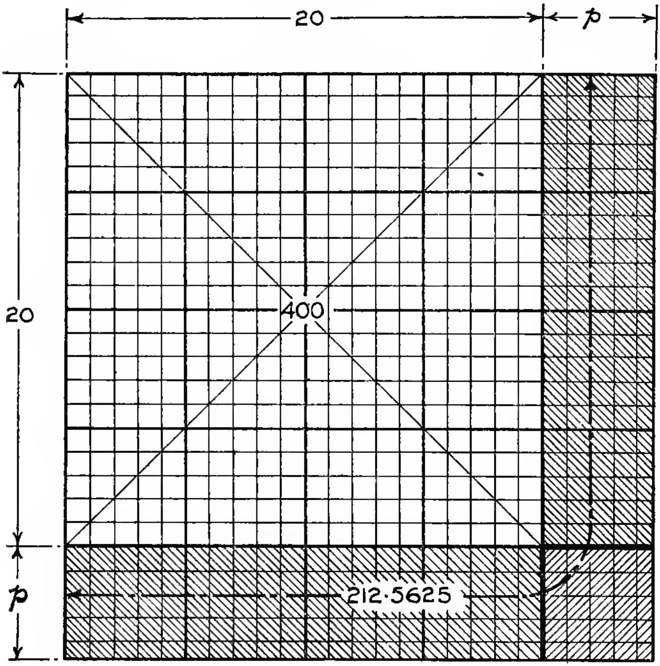

FIG. 35.

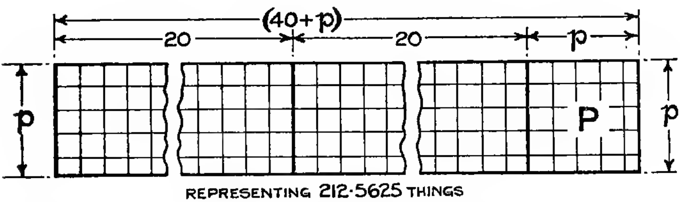

FIG. 36.

Now, rearranging these three subgroups contiguously, to form a rectangular group, as in Fig. 36, the total number

of rows is clearly $\{(20 \times 2) + p\} = (40 + p)$; and the number of things in each row is p throughout.

We ascertain the largest whole number of things per row in this assemblage by dividing 212·5625 by (20×2), bearing in mind the fact that provision must be made for the square subgroup P in Fig. 36. Representing this number by x, we proceed to withdraw $(40 + x)$ rows with x things in each row—*i.e.* $(40x + x^2)$ things—from the assemblage, first ascertaining that the maximum permissible

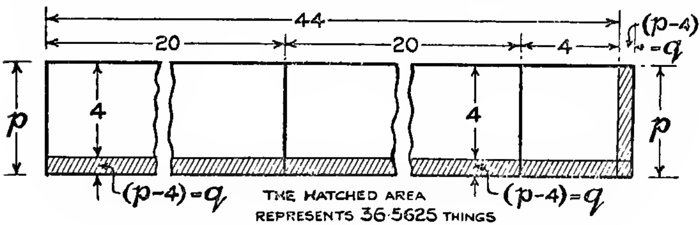

FIG. 37.

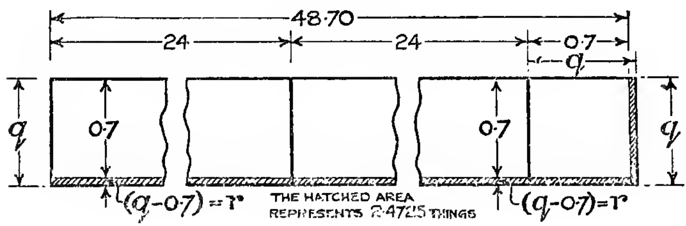

FIG. 38.

magnitude of x is 4·00. Hence, we subtract $\{(4 \times (40 + 4)\} = (4 \times 44) = 176$ things, as shown in the calculation.

This leaves 36·5625 things, arranged as shown by the hatched portion of Fig. 37.

This remaining assemblage, being of uniform width—*i.e.* $(p - 4) = q$—throughout, may be rearranged to form a rectangular group, as in Fig. 38. It will be seen to comprise $(48 + q)$ rows, with q things in each row.

(NOTE.—*These diagrams are drawn to successively larger scales, to secure clearness in the illustrations without their*

becoming unwieldy in size. The student is advised to draw them all for himself, correctly to scale, for this and other similar cases.)

The largest magnitude which may be provisionally assigned to q without exceeding the total number of things (viz. 36·5625) comprised in this remaining assemblage is 0·70; and hence we subtract $\{0{\cdot}70 \times (48{\cdot}00 + 0{\cdot}70)\} = (0{\cdot}70 \times 48{\cdot}70) = 34{\cdot}09$ things, as shown in the calculation.

This further withdrawal from the assemblage leaves 2·4725 things, arranged as indicated by the hatched portion of Fig. 38; and this remaining assemblage may be rearranged to form a rectangular group, as in Fig. 39. It will be seen to comprise $(49{\cdot}40 + r)$ rows, with r things in each row.

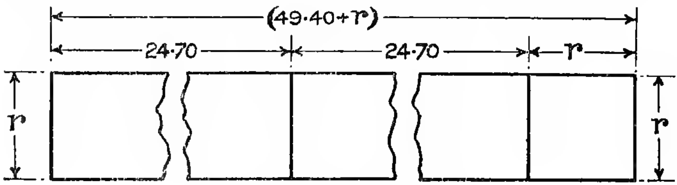

FIG. 39.

A little consideration will show that, by assigning to r the magnitude 0·05, the whole number is accounted for exactly; and the arrangement of the 612·5625 things in a square group has been effected, there being 24·75 rows with 24·75 things in each row.

Now let us follow this through finally, without presupposing any square group arrangement of the given number as was assumed in preparing the sketch for Fig. 35.

The 612·5625 things being supposed to lie, in no particular order or arrangement, on a table, 400 are withdrawn and arranged in a square group, as indicated in Fig. 40. Of the remaining 212·5625 things we take 176, and dispose them along two adjacent sides of the 400 square group—80 (in 4 rows of 20) along the right-hand side, 80 along the

bottom, and 16 at the right-hand bottom corner, giving again a square group, with 24 rows, and 24 things in each row, as in Fig. 40.

We have 36·5625 things left, of which we take 34·09, and dispose them along two adjacent sides of our square group as before.

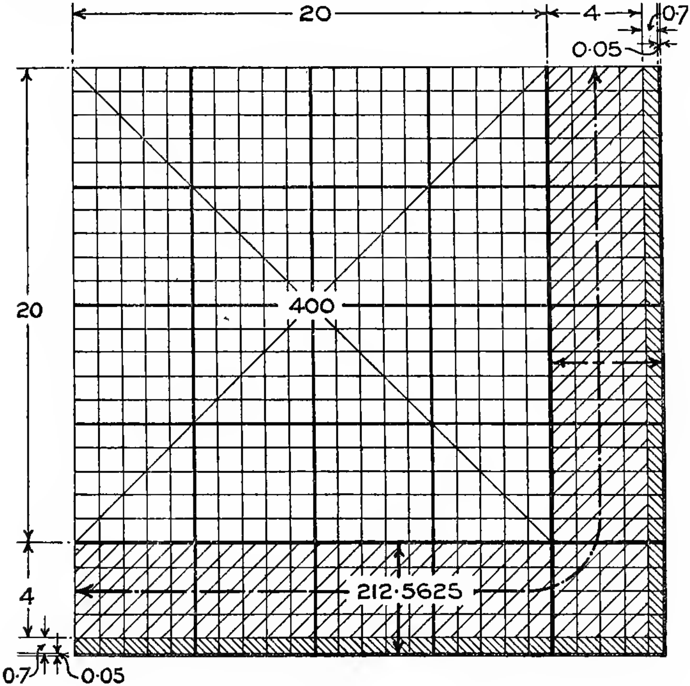

Fig. 40.

(Note.—*We need not take the same two sides as before, nor any two particular sides, so long as they be adjacent; but it is obviously more convenient to follow a settled order or procedure.*)

Placing 16·8 of these things—in 24 rows of 0·7—along the right-hand side, 16·8 along the bottom, and 0·49 at the right-hand bottom corner, a square group (comprising

24·7 rows with 24·7 things in each row) is obtained, as in Fig. 40.

Lastly, we take the remaining 2·4725 things, placing 1·235 of them (in 24·7 rows with 0·05 things in each row) along the right-hand side of our penultimate square group, 1·235 similarly along the bottom, and 0·0025 at the right-hand bottom corner, thus completing the required arrangement, as indicated in Fig. 40.

Thus, by combining logical argument with actual arrangement and rearrangement of things, square root is seen to be no more than a comparatively simple—and wholly physical—operation performed upon real things forming a stationary group—*i.e.* they are not passing from one group to another during the operation. This latter observation is extremely important, as will be seen presently.

In the example chosen above a complete determination was obtained at the second decimal division—though, of course, some of the things would need to be cut into 200 equal parts. In those cases which do not admit of so early a solution as that chosen, and in those which do not admit of any exact solution at all, the determination of the square root to any specified degree of approximation follows clearly on the lines indicated, and will need no further discussion here.

Simplified Methods for Square and Cube Roots.—In practical calculations a large proportion of the determination of square and cube roots is concerned with comparatively small numbers ; and frequently the result is required to be of only such a degree of approximation as will permit the selection of some "stock" size for a piece. For example, the depth of a beam need only be determined to the nearest inch, as a rule ; the diameter of a shaft to, at most, the nearest quarter of an inch ; and so on. In such cases the square roots may be readily obtained, with sufficient accuracy, by means of a simple method which is based upon the fact that, if b be small as compared with a, then $(a \pm b)^2 = a^2 \pm 2ab$ (very nearly), for b^2 will, under

such conditions, be very small as compared with $(a^2 \pm 2ab)$—as will be clearly seen on reference to Fig. 41, which is drawn for the ratio $b : a = 1 : 10$. Obviously, to spread the appropriate portion of the small hatched group (which represents b^2) along two adjacent sides of the square group would have an exceedingly small effect upon the length and breadth (*i.e.* the square root) of the assemblage.

Speaking broadly, the method as described below will

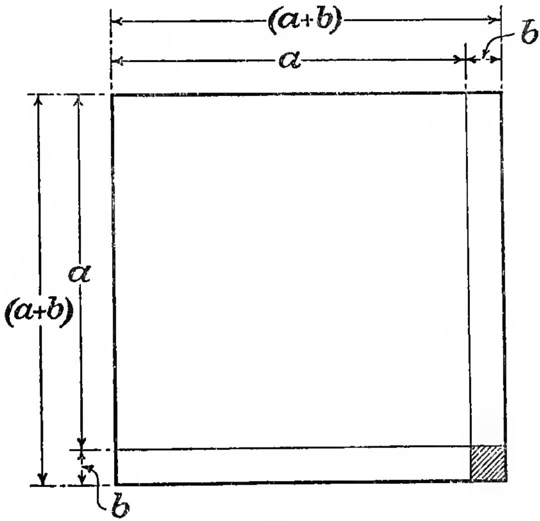

FIG. 41.

be found both rapid and convenient in use ; and it possesses the further advantage of being, after a little practice, capable of application mentally to many cases.

The square root of every number consists of two parts—one part a whole number, and the other part a fraction—added together. In the case of a number less than 1, the integral part of the square root is 0 ; and with a number which is the square of an integer, the fractional part of the root disappears. These special cases do not affect the general principle, however.

If the square root of a given number be required we may regard the given number as consisting of $(a^2 \pm 2ab)$, which, in turn, we may consider as equal to $(a \pm b)^2$, where a is an integer and b a fraction. The magnitude of a may be determined by inspection; for a^2 is that "square-of-an-integer" which differs from the given number by less than does any other square-of-an-integer. The difference between a^2 and the given number is regarded as being equivalent to $2ab$; and hence b will be found as the quotient obtained from dividing this difference by $2a$.

When a^2 is less than the given number, the square root will be $(a+b)$; and when a^2 is more than the given number, the root will be $(a-b)$.

The application of the method will be shown best by means of a few typical examples:

EXAMPLE IV.—*To find the Square Root of* 26·42

The nearest square of an integer being 25, a will be 5.

The difference between 26·42 and 25 is $+1{\cdot}42$, which we regard as equivalent to $2ab$. But $2ab = 2 \times 5 \times b = 10b$; and hence $10b = 1{\cdot}42$; so that $b = 0{\cdot}142$.

Therefore, by this method, $\sqrt{26{\cdot}42} = 5 + 0{\cdot}142 = 5{\cdot}142$.

Determined by the ordinary method, $\sqrt{26{\cdot}42} = 5{\cdot}14003 \ldots$.

EXAMPLE V.—*To determine* $\sqrt{51{\cdot}27}$

Here a^2 is obviously 49; and $a = 7$.

The difference is $51{\cdot}27 - 49 = +2{\cdot}27$; whence $2ab = 14b = 2{\cdot}27$; and $b = 2{\cdot}27 \div 14 = 0{\cdot}162 \ldots$.

Therefore, by this method, $\sqrt{51{\cdot}27} = 7 + 0{\cdot}162, \ldots = 7{\cdot}162 \ldots$.

By the more accurate methods, $\sqrt{51{\cdot}27} = 7{\cdot}1603 \ldots$.

EXAMPLE VI.—*To determine* $\sqrt{119{\cdot}2}$

In this case it is clearly well to work on the basis of $(a-b)^2$; and hence $a^2 = 121$, giving $a = 11$.

The difference is $119{\cdot}2 - 121 = -1{\cdot}8$; whence $2ab = 22b = -1{\cdot}8$; and $b = -0{\cdot}082$.

Therefore, by this method, $\sqrt{119{\cdot}2} = 11 - 0{\cdot}082 = 10{\cdot}918$.

By the more accurate methods, $\sqrt{119{\cdot}2} = 10{\cdot}9179 \ldots$.

EXAMPLE VII.—*To determine* $\sqrt{93 \cdot 2}$

This example introduces a case in which the given number lies about midway between the squares of two consecutive integers, and in such cases (for the purpose of complying with the condition that b shall be as small as possible in comparison with a) it is well to take a as the mean of the two integers.

The given number lying between 81 and 100, let us take a as 9·5, giving $a^2 = 90 \cdot 25$.

Then the difference is $93 \cdot 2 - 90 \cdot 25 = +2 \cdot 95$; and hence $2ab = 19b = +2 \cdot 95$; giving $b = +2 \cdot 95 \div 19 = +0 \cdot 1553$.

Therefore, by this method, $\sqrt{93 \cdot 2} = 9 \cdot 5 + 0 \cdot 1553 = 9 \cdot 6553$.

By the more accurate methods, $\sqrt{93 \cdot 2} = 9 \cdot 654 \ldots$

EXAMPLE VIII.—*To determine* $\sqrt{131 \cdot 6}$

For the same reasons as in Example VII., take $a = 11 \cdot 5$, giving $a^2 = 132 \cdot 25$.

The difference, then, is $131 \cdot 6 - 132 \cdot 25 = -0 \cdot 65$; whence $2ab = 23b = -0 \cdot 65$; and $b = -0 \cdot 65 \div 23 = -0 \cdot 028 \ldots$

Therefore, by this method, $\sqrt{131 \cdot 6} = 11 \cdot 5 - 0 \cdot 028 = 11 \cdot 472$.

By the more accurate methods, $\sqrt{131 \cdot 6} = 11 \cdot 4717$.

To square such numbers as 9·5 and 11·5 mentally is, of course, a simple matter on the basis of $(a+b)^2 = a^2 + 2ab + b^2$. Taking the latter case for example :

$$\begin{aligned}(11 \cdot 5)^2 &= (11 + 0 \cdot 5)^2 = 11^2 + 2(0 \cdot 5 \times 11) + (0 \cdot 5)^2 \\ &= 121 + 11 + 0 \cdot 25 = 132 \cdot 25.\end{aligned}$$

Many such squares will, with a little practice, be found to remain in the memory, and will present themselves for use instantly as occasion arises.

Given a clear appreciation and a little practice, the foregoing method will be found easy to apply by mental process without calculation on paper, enabling one to write down square roots of even fairly complicated numbers immediately, with sufficient accuracy for practical purposes.

With slight modification, the method may sometimes be applied to large numbers, as will be seen from the following example :

EXAMPLE IX.—*To determine* $\sqrt{161327}$

In such cases we allow b to consist of a mixed number—partly integral and partly a fraction—taking a as merely the most convenient integer. If handled with reasonable care there is little risk of serious error in this, for b will still be quite small in comparison with a.

Noticing that the given number is not much in excess of 160000, we take $a = 400$, giving $a^2 = 160000$.

Then the difference is $161327 - 160000 = +1327$; whence $2ab = 800b = +1327$; and $b = 1327 \div 800 = 13{\cdot}27 \div 8 = 1{\cdot}66$.

Therefore, by this method, $\sqrt{161327} = 400 + 1{\cdot}66 = 401{\cdot}66$.

By the more accurate methods, $\sqrt{161327} = 401{\cdot}6553\ldots$

Of course, where much manipulation is required, it will probably be found both quicker and safer to use the ordinary arithmetical method, which may be made extremely simple and rapid by abbreviations on the lines of the well-known "Italian" method of abbreviated division.

Here a word with regard to abbreviated and "short" methods generally may be of service. All methods and means for minimising purely mechanical work in calculations should be cultivated, used and developed—where they prove, on fair trial, to effect their purpose. It is, however, quite easy to misapply such methods, and also—mainly through excessive admiration for them—to so complicate them that they occupy actually more time in operation than the (so-called) long methods. Moreover, the minds of men are individualistic to a high degree—probably no less so than the bodies of men; and just as a batting stroke at cricket which may be enormously effective to a long-armed man may be disastrous if attempted by a short-armed man of equal ability, so a mathematical method which suits one man admirably may be totally unsuitable to another. Each must develop his own equipment for himself—taking care, of course, to profit in all possible ways by the experience of others, and trying a suggested method very thoroughly before definitely abandoning it as useless. Also, it is necessary to observe carefully the scope and limitations of each particular

method; and to employ each part of one's equipment only in circumstances which render its use properly permissible and advantageous. The aim should be to acquire as comprehensive an equipment as possible, having regard to the work with which one is more particularly concerned; every item should have its own special purpose, *and be kept in frequent use*—otherwise it is probably crowding out some more valuable item. For instance, a carpenter's tool-chest which contains all sorts of weird implements which one never uses is more of a hindrance than a help; but, on the other hand, a single chisel, while a most useful tool, will make a poor show when it comes to plane up a floor board or drill a $\frac{1}{2}$-in. diameter hole. Really, as has been remarked several times already in the preceding pages, the whole matter reduces itself to the necessity for each man to develop his own personality in every way to the best advantage.

A method similar to that described above for square root may be applied in many cases to cube root.

By similar argument, if b be small in comparison with a, there may be little error in assuming that, for practical purposes, $(a \pm b)^3 = a^3 \pm 3a^2b$.

The student should demonstrate for himself, by means of a sketch on the lines of Fig. 85, the soundness of the argument in this case; and also practical limits within which it may be more or less safely applied.

Then, desiring to determine the cube root of a given number, the nearest cube-of-an-integer is seen by inspection; and thus the magnitude of a is ascertained. The difference between the given number and a^3 is regarded as equivalent to $3a^2b$; and hence the magnitude of b may be found as the quotient obtained on dividing the difference by $3a^2$. When a^3 is less than the given number, the cube root will be $(a+b)$; and when a^3 is more than the given number, the cube root will be $(a-b)$.

Two typical examples will doubtless suffice to show the application of the method to practical cases.

EXAMPLE X.—*To determine* $\sqrt[3]{70\cdot83}$

Clearly 64 is the nearest cube-of-an-integer, and thus, $a=4$.

The difference being $70\cdot83-64=+6\cdot83$, it follows that $3a^2b=3\times16\times b=48b=+6\cdot83$; whence, $b=+6\cdot83\div48=+0\cdot142....$

Therefore, by this method, $\sqrt[3]{70\cdot83}=4+0\cdot142,...=4\cdot142....$

By the more accurate (but also more troublesome) methods, $\sqrt[3]{70\cdot83}=4\cdot137....$

EXAMPLE XI.—*To determine* $\sqrt[3]{337\cdot5}$

Here the nearest cube-of-an-integer is 343, giving $a=7$.

The difference, then, is $337\cdot5-343=-5\cdot5$; and hence $3a^2b=3\times49\times b=147b=-5\cdot5$; so that $b=-5\cdot5\div147=-0\cdot0374....$

Therefore, by this method, $\sqrt[3]{337\cdot5}=7-0\cdot0374=6\cdot9626$.

By the more accurate method, $\sqrt[3]{337\cdot5}=6\cdot963$.

Bearing in mind the fact that, for cube and higher roots, the only practicable method is by the use of logarithms, it will be seen that the above-described method, with all its faults, may be very useful on occasion when tables of logarithms are not available.

Powers and Roots of Negative Quantities.—We arrive now at a point which is of great importance; and which gives rise to much confusion and trouble, simply because it is seldom considered from the standpoint of actual fact.

It is often stated that "the square of a negative quantity is positive; the cube of a negative quantity is negative; and so on." Probably every student of mathematics has at one time or another asked himself the question as to why this should be so.

Again, it is commonly believed that "we cannot extract the square root of a negative quantity"; and again the question which naturally arises is—why not?

Usually the "rule" is accepted with more or less docility, and the knack of manipulating the written symbols

acquired; but the question is seldom answered satisfactorily—with the consequence that there is always a feeling of insecurity or distrust when the need arises for applying the processes to actual work in the affairs of everyday life.

The matter is really very simple, as will be seen presently; and such difficulty as arises is due entirely to a loose usage of the symbolism, combined with omission of essential parts of the argument.

We have to explain how it is that $(-x)^2 = +x^2$. The explanation for other indices will follow naturally from this.

If we owe a man 4 things, the fact might be recorded in accounts as (-4); and we have seen that "raising to a power" is merely a matter of repetition (by addition) of the unit thing. Now, it is not easy to see how repeating a debt (*i.e.* adding to it) can turn it into an asset—and were such an operation possible, it would have distinct disadvantages as well as advantages. Of course this is not the fact at all.

Let us briefly review the inferences already drawn above regarding Involution; those regarding "Negative Quantities" (Chapter II.); and those regarding the "Rule of Signs" (Chapter II.).

By (x^4) we mean an assemblage consisting of an original unit thing repeated to (x^4) times. We have not raised "*x things*" to the "*fourth power*"—whatever that may mean.

Also, by "-5" we mean one real thing which has the effect of reducing our assets, repeated to 5 times.

It has been shown (Chapter II.) that the question of "positive nature" and "negative nature" arises only when we are considering the effects (upon groups of things) of things passing (or required to pass) into or out of those groups; and in Involution this passage of things does not concern us, for we do but count the things which form the final assemblages, whatever their nature.

The question is similar to that raised and discussed under the heading "Manipulation of Positive Integral Indices"

(pp. 68 *et seq.*); with the slight extension occasioned by the introduction of the negative sign.

By $(-x)$ we mean one original unit real thing (whatever its nature) repeated to $(-x)$ times—*i.e.* x of such things withdrawn from a group. So here, as in all similar previous cases, x denotes a number of *actions*; and the negative sign denotes the nature of these actions—*i.e.* withdrawals from a group.

We might, of course, write $(-x)$ as $(-x)^1$ without altering its effect; but in so doing it should be noticed that while the symbolism is precisely that of Involution, the operation implied is not really Involution.

The action in Involution is that of repeating things from a store, so that the things accumulate to form a group upon which our attention is focussed, to the exclusion of the effects upon the store (so long as it is not exhausted) from which the things are obtained; and whether these accumulated things be regarded as forming a group in themselves, or are considered as additions to some other group, the counting of them will be the same.

On the other hand—*i.e.* when dealing with $(-x)^1$—we cannot withdraw things, focussing attention upon the effects of their withdrawal, unless there be given a group from which the things may be withdrawn.

Similarly, the symbol $(-x)^2$ does not really imply the process of Involution. The statement $+(-x)^2 = +x^2$ holds only if the intermediate step be such as may be recorded thus:

$$+(-x)^2 = +\{(-x) \times (-x)\} = +x^2.$$

In this, however, as is shown in Fig. 1 (p. 36) and the discussion on the "Rule of Signs" to which it relates, the two negative signs in the intermediate step have different meanings—one denoting the *action of withdrawal* (or, in the language of accountancy, *discounting*, and the other denoting the *nature of the things withdrawn* (i.e. *bills*, or *asset reducers*).

The statement might be interpreted thus: *With regard to a group comprising assets and liabilities in a certain kind of thing, on x occasions a bill demanding payment of x things* (i.e. *a bill demanding payment of one thing repeated to x times*) *is withdrawn* (*or completely discounted*), *leaving the balance of assets over liabilities increased by x^2 things.*

It will be seen that the group must comprise liabilities as well as assets; for if there were no bills none could be discounted. An equivalent effect might be produced by the rendering of a voucher or credit-note; but this would not introduce the negative sign at all in the records.

Let us consider an arithmetical example, to show the point more clearly.

Suppose a man has assets amounting to £140; and liabilities amounting to £40, consisting of 10 bills, each demanding payment of £4.

His state of affairs at the moment might be recorded as

$$\{140+10(-4)\}=140-40=£100.$$

If now, for some reason, four of these £4 bills are completely discounted, the new state of the man's affairs might be recorded as

$$\begin{aligned}[140+10(-4)+\{(-4)\times(-4)\}]&=\{140+10(-4)+(-4)^2\}\\&=140-40+16=£116;\end{aligned}$$

showing the balance of assets over liabilities 16 (*i.e.* 4^2) more than before.

The student may be left to build up for himself, on these lines, the arguments explaining $(-x)^3$, $(-x)^4$, etc.

For the "square root of a negative quantity" the explanation is even less difficult—though perhaps, to the student who has been trained on strictly orthodox lines, it may at first sight appear far more startling—than the foregoing in respect to the "square of a negative quantity." The difficulty which has arisen is, however, in this case as in that, due merely to looseness in using the symbolism, and omission of essential parts of the argument.

Let us suppose that we have 25 bills, each demanding payment of one thing. The effect of these bills *upon our group of assets* might properly be denoted as "-25."

Now, it has been shown above that to extract the square root of a number we have only to arrange that number of things in a square group; and there is clearly no difficulty whatever in so arranging 25 (or any other number of) bills.

On this basis, then, obviously $\sqrt{-25} = -5$; for the bills must remain bills throughout the process of their arrangement or rearrangement.

Here, again, the things do not pass either into or out of the group during the process of arrangement. The total number of things comprised in the group is not altered in any way; and in taking the square root we do but count the number of things in each row of the square-group arrangement.

The *nature* of the things does not affect the question in any way, since the things have merely to be arranged in a particular way, and counted. Moreover (and in this lies the fallacy), the negative sign bears no relation to the nature of the things themselves, apart from other things. The 25 things concerned are 25 real things; and the negative sign indicates neither more nor less than that they are required to pass out of some asset-group of similar things. If the 25 things be pounds sterling, they represent, with the negative sign prefixed, a debt of £25. If the debtor has assets sufficient to meet and discharge the debt, he may take the 25 sovereigns or notes in his hand, lay them on a table, and find their square root by arranging them in a square group—*and they will still be a debt*, of the same amount, until they are actually paid away. Further, they must remain *real pounds* when they have been paid away, or there will be trouble.

When we say that $\sqrt{+x^2} = \pm x$, we are really not referring to the actual operation of Evolution at all. We are only stating *in reversed order* (always a more or less dangerous proceeding) the first and last steps of a quite different process.

It is shown, in the discussion on the "Rule of Signs" (Chapter II.), that

$$+\{(+x)\times(+x)\} = +x^2;$$

and also that

$$+\{(-x)\times(-x)\} = +x^2.$$

The symbolism of Evolution has been loosely applied to this, on the fallacious assumption that in each of these equations we may extract the square root of both sides, without affecting the equality. Obviously, this assumption is entirely unjustifiable; for we have seen that the sign " = " in these cases does not mean complete similarity, but only that one operation is *as good as* another of a different kind (see pp. 32 *et seq.*); or that, though different in kind, the two operations will produce an equivalent effect upon the *assets in some other group*.

In the statement $+\{(-x)\times(-x)\}$, we have seen that one of the x's denotes *actions*, and the other *things*; while one of the negative signs particularises the actions as withdrawals from some group, and the other indicates that the things have the effect of reducing the assets of *that particular group*—though of that group alone. It is manifestly impossible to arrange such a combination in a square group —which is all that extracting the square root can mean in physical fact—except on the basis described above with regard to the debt of £25; *i.e.* by regarding them as real things, without regard to their influence upon any other groups, and arranging them accordingly for counting.

On this fallacious assumption, of course, it follows that $\sqrt{-x^2}$ has no physical counterpart; and the device of restating it in the forms:

$$\sqrt{(+x^2)(-1)} = \pm x\sqrt{(-1)} = \pm xi,$$

does not help very much from the practical point of view.

However, since it is the assumption which is fallacious, there is little need for the practical man to be either despondent or perturbed at the derivation of a fallacious inference

from it. Even were a process available by which debts could be made "imaginary" by mere juggling with written symbols, it might prove a by no means unmixed blessing; for our debtors might be more skilful in applying the process than we. Fortunately, however, there is no such process; nor is one likely to be devised.

There is, in the foregoing discussion, not the slightest desire to upset orthodox views, nor to interfere with them in any way. The effects of the inferences drawn, with regard to Quadratic Equations and similar matters, are shown and explained in Chapters VIII. and IX.; but in the meantime, the practical student may rest assured that if some investigation relating to matters of physical fact lead to the supposition that the square root of a negative quantity (on the ordinary basis of mathematical symbolism) is required, he should immediately review the *investigation*; for at some stage or another of that investigation he has imposed some condition which no ordinary human being can fulfil—such as the accommodation of a quart in a pint pot at one operation.

CHAPTER V

INDICES AND LOGARITHMS

Indices Generally.—As a rule, Indices are regarded as "Power Indicators"; and provided there be no vagueness as to their actual meaning with regard to real things, there is no harm in adopting this brief description for them. If the student has thoroughly grasped the arguments outlined in Chapters II., III. and IV., he will find little difficulty in interpreting correctly the various statements which arise in the manipulation of Indices for practical purposes.

Apart from the few typical instances in which an obvious simplification may be effected by manipulating Indices in accordance with one or more of the "Laws" (see p. 68), it is often a matter of some difficulty to understand why such manipulation should have been extended to cover so wide a field as it is made to cover in the usual treatment of mathematics. There is, however, a good reason for much of this extension; and the inquiring student may derive beneficial reassurance from a consideration of the matter.

It is shown, in the discussion on Evolution (Chapter IV.), that the extraction of roots generally (or, at least, those other than the square root) would be at best a very tedious business—while many would be altogether impracticable—without the use of logarithms. Since logarithms are merely Indices, and in order that they may be tabulated in a form at once sufficiently concise and sufficiently comprehensive to cover conveniently the whole range of ordinary practical calculations, it follows that the student

who would use logarithms effectively must be able to manipulate Indices over a correspondingly wide field. It is, perhaps, not absolutely necessary that a man who *uses* logarithms should be able to *calculate* them—though that ability is well worth acquiring, and (except that it is somewhat tedious, and calls for the exercise of care) presents no great difficulty. He must, however, be able to manipulate them properly; and for this he needs a knowledge and understanding of their influences and effects—particularly in combination among themselves.

For this purpose, it has been found desirable and convenient to think of *all* Indices—positive and negative, integral and fractional—as "Power Indicators," regardless of the fact that a fractional index denotes a *root*, which is distinguished from a *power* (more properly indicated by an integral index) by classifying the one under the heading of Evolution and the other under the heading of Involution; and regardless, also, of the fact that a negative Index must obviously indicate an operation different from that indicated by a positive Index.

Clearly, so long as the danger of vagueness or misunderstanding is prevented by a full comprehension of the facts, there can be no harm in such modifications of our symbolism; and it would be foolish to deny ourselves the use of a highly convenient and helpful device for no better reason than because it involves a risk which we have already fully appreciated and adequately provided for.

Positive integral indices have been considered (in Chapter IV.) at sufficient length to render further discussion unnecessary. It remains, therefore, to consider negative and fractional indices, both separately and in combination with each other, as well as in combination with positive integral indices.

Negative Integral Indices.—The symbol x^{-2} permits of a very simple explanation if it be regarded from the standpoint of its influence upon some real thing as a factor of repetition—the repetition being of a negative character.

Positive repetition having the effect of increasing, it is but logical to suppose that negative repetition has the effect of reducing; and as positive repetition operates to *build up* on the basis of a single unit thing, it is not surprising to find that negative repetition operates to *break down* a single unit thing into parts. The inference will be perfectly plain if the arguments be applied to the case of some assemblage consisting of things, in which each " thing " is itself (as it always is in practical work) an assemblage of smaller things —or *parts*.

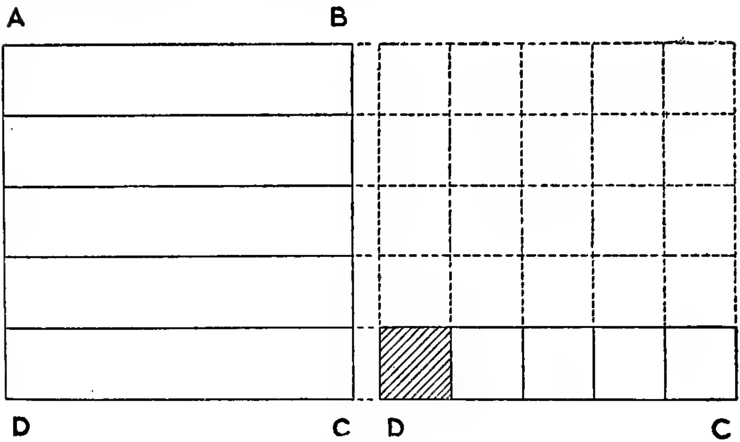

FIG. 42.

Just as we have seen that x^2 (or, if it be preferred, x^{+2}) means that an original unit thing is first repeated to x times to form a row, and then each thing in that row repeated to x times to form a layer, so x^{-2} means that an original unit thing is first divided into x equal parts (this process being recorded as x^{-1}), and then each of those parts divided into x equal parts; the symbol x^{-2} indicating the magnitude of the final result with relation to the original unit thing.

By way of an example, if x be 5, x^{-2} becomes 5^{-2}, and the operation is illustrated in Fig. 42; the original unit thing

being represented by the square ABCD, and the magnitude of the final result being obviously $\frac{1}{25}$ of the original unit thing.

In Fig. 42, the second division into x equal parts is made at right angles with the first. There is, of course, no real need for this procedure; nor for representing the original unit thing by a square. We might represent the original unit thing by a rectangular strip, as in Fig. 43; making the first division transversely into 5 equal parts to give the shorter rectangle indicated at (a) in the same illustration, and the final division parallel with the first, to give the result as indicated at (b). If the area of the original rectangle in Fig. 43 be equal to that of the square in Fig. 42, the final results will clearly be equal in area for both cases.

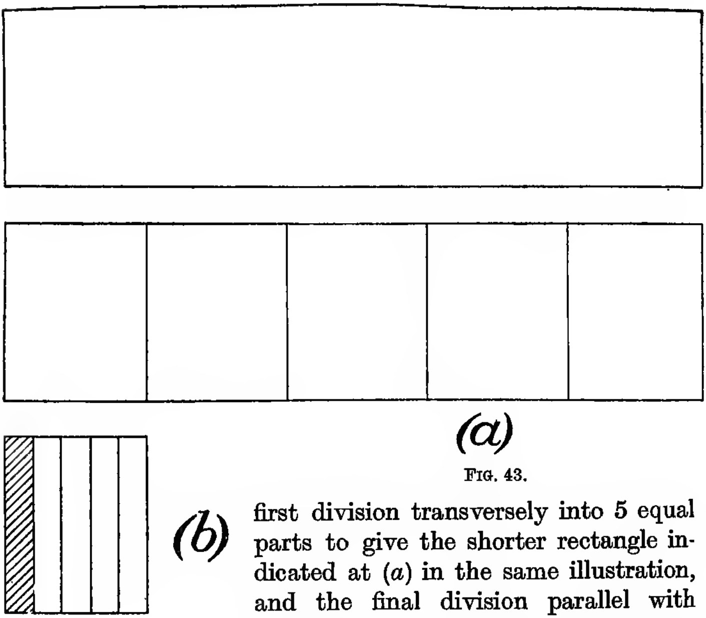

FIG. 43.

For the purposes of clearness in illustration, however, and also for convenience in the interpretation of negative indices of higher degrees on an orderly and easily intelligible basis, such a method as that followed in Fig. 42 is obviously preferable.

To interpret x^{-3}, it is well to represent the original unit thing by a cube; and to make the successive subdivisions in directions parallel with the edges of this cube.

For the case of x^{-4}, we might conveniently represent the original unit thing by a prism, x^2 in length, x in width and x in height, again making the successive subdivisions parallel with the edges of this symbolical solid—always remembering, of course, that it represents either an assemblage of things, or a single thing which itself consists of an assemblage of smaller things or parts.

Taking the unit thing as represented by the prism indicated in Fig. 44, the first step would be to divide it into x equal parts, giving the cube indicated at (a) in the same sketch. The division of this into x equal parts would give the slab indicated at (b); the division of this into x equal parts would give the rod indicated at (c); and finally, the division of this rod into x equal parts would give the small cube—the result of the complete operation—indicated at (d) in Fig. 44.

Obviously, the operation indicated by x^{-2} and illustrated in Fig. 42 might equally well be indicated by $\left(\frac{1}{x^2}\right)$; and in much ordinary work—particularly in calculations dealing with fractions where factors common to the numerator and denominator may be cancelled—this is the more convenient of the two forms. The significance being the same, however, it is well to fix the equivalence of the two symbols in the mind; and this may be done by recording the fact thus:

$$\left(\frac{1}{x^2}\right)=x^{-2}; \ \left(\frac{1}{x^3}\right)=x^{-3}; \text{ and, generally, } \left(\frac{1}{x^n}\right)=x^{-n}.$$

Even in using logarithms, the result is, of course, the same whichever method of writing be followed; for, clearly, adding two negative things has precisely the same effect as subtracting two positive things of equal magnitude.

The negative index does, however, serve a highly useful

purpose in showing very clearly that logarithms of numbers range from very large negative numbers to very large

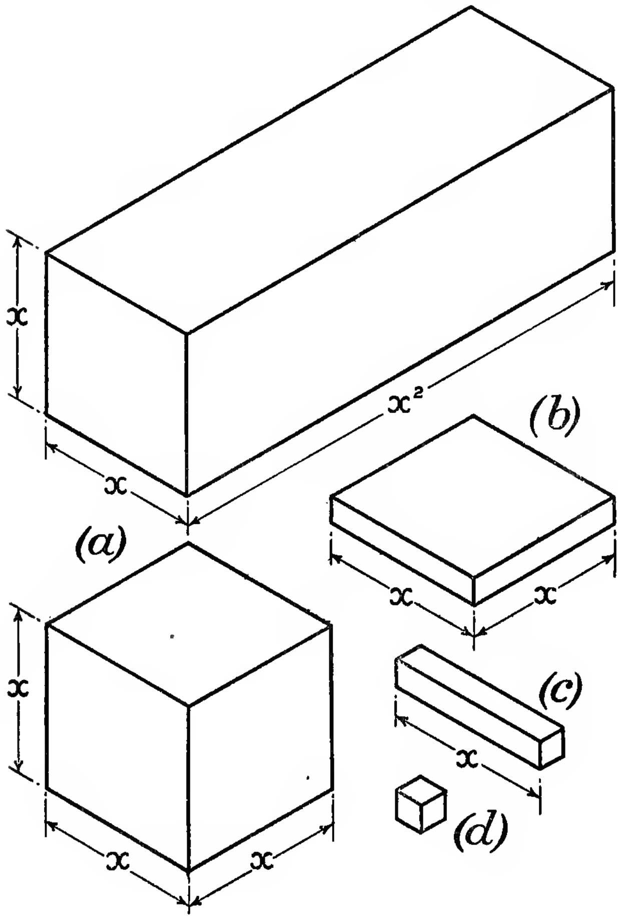

FIG. 44.

positive numbers as the numbers to which they relate vary from extreme smallness to extreme greatness.

In text-books, as a rule, the method adopted for proving that $\frac{1}{x^2}=x^{-2}$ is to replace the 1 forming the numerator of the fraction by x^0; whence, the statement becomes,

$$\frac{1}{x^2}=\frac{x^0}{x^2}=x^{0-2}=x^{-2}.$$

It having been shown (p. 68) that x^0 represents *one unit thing*, this statement may be made more simply by writing in the words; but this may be left to the student to elaborate for himself if necessary.

Fractional Indices.—It is stated, on p. 77, that $\sqrt[2]{x^3}$ might be written more conveniently as $x^{\frac{3}{2}}$, or $x^{1\cdot5}$.

As with negative indices, the convenience of indicating a root as though it were a fractional power lies solely in the consistency of the symbolism with the principle of logarithms which is thereby obtained; but as the aid of logarithms is much more generally necessary in the extraction of roots than in the determination of magnitudes or numbers represented by negative integral indices, the convenience of fractional indices is correspondingly greater than that of negative integral indices.

Let us consider first the question as to why $\sqrt[2]{x^1}$ may be denoted as $x^{\frac{1}{2}}$; for this is commonly a cause of difficulty and vagueness.

The symbol $x^{\frac{1}{2}}$ is clearly at variance with the ordinary interpretation of Involution on the basis of physical fact; for we should find some difficulty in performing, with real things, the operation of "multiplying a number by itself half a time." Nor can we say that $x^{\frac{1}{2}}$ implies that a unit thing is to be repeated to x times on "half an occasion" or in "half a direction." Yet we could show quite easily, by arranging and rearranging real things on the lines of Figs. 33 and 34 (pp. 76 *et seq.*), that $\sqrt[2]{x^2}=x^1$; that $\sqrt[2]{x^4}=x^2$; and so on. And since 1 is the half of 2, and 2 the half of 4, there would seem to be some reasonableness in supposing that $\sqrt[2]{x^1}$ should follow a similar order.

Now, it should be observed that " $\sqrt{x^2}$ " is but a symbol which men have devised, and adopted by common consent, to represent a certain process or operation which they found themselves capable of performing with real things; and it was only because their work imposed upon them the necessity of performing the operation that they proceeded to discuss it—thus introducing the need for a name by which they could speak of it, and a convenient symbol with which they could indicate it in writing. There is no actual virtue possessed by the symbol which *makes* it definitely indicative of any process or thing whatsoever; nor have we any reason to suppose that it is supernaturally endowed with any property by which it can govern real things. When man encounters a new experience, he devises a name for it; but it does not follow that by devising a new word (by way of a name) he can assure himself of encountering a corresponding experience.

There is, however, this important fact to notice. If we have a set of things (or experiences) which are connected by a definite order of procedure, and we seek to devise a scheme of symbolism to represent those things, it is but reasonable that we should ensure that the symbols shall follow—and indicate as clearly as possible—the order of procedure which connects the things symbolised. The fact that the symbols are then found to follow an order (or a "law") does not, of course, justify us in assuming that this law will govern the real things over an indefinitely extensive range, for the symbols follow the law only because we so devised them that they should; but at the same time, an examination of the law as represented by the symbols may reveal things which, though connected with the other things by the same order of procedure—and, therefore, properly belonging to the "set" for which the symbols were devised—may have escaped our observation in the first survey of the "set." If, on examination in the light of actual fact, these things indicated are found to

be *not* subservient to the law, then, clearly, the law has either been erroneously framed, or it has limitations. If, however, it be found that the things *are* subservient to the law, we are bound to admit them as belonging to the "set" of things symbolised. It is on this basis that the physical interpretation of the statement $\sqrt{x}=x^{\frac{1}{2}}$ may be found.

From the standpoint of Evolution, the symbol $\sqrt{x}$ has been shown to denote a process of simple rearrangement and counting of the things forming a group. Similarly, it has been shown that $\sqrt[2]{4^3}=8$; and by reversing the physical operation it could be shown that $\sqrt[3]{8^2}=4$. Combining this with the facts that $\sqrt[2]{x^4}=x^2(=x^{\frac{4}{2}})$; and $\sqrt[2]{x^2}=x^1(=x^{\frac{2}{2}})$; we are led to the inquiry as to whether $\sqrt[2]{4^3}$ may not properly be written as $4^{\frac{3}{2}}$; $\sqrt[3]{8^2}$ as $8^{\frac{2}{3}}$; and, generally, $\sqrt[q]{x^p}$ as $x^{\frac{p}{q}}$. If so, then it follows that there must be a physical reality corresponding to the statement $\sqrt[2]{x^1}=x^{\frac{1}{2}}$; and if we can find this physical reality, the inquiry will be satisfactorily answered.

Let us suppose that $\sqrt[2]{x^1}$ means, in accordance with our basis of Involution, that an original unit thing is repeated to some number (say n) of times to form a row. On this basis, we may write $\sqrt[2]{x^1}=n$. Now, if we knew n, we might argue that, the unit thing having been repeated to n times to form a row, if each thing in the row be repeated to n times to form a layer, the resulting assemblage would comprise n^2 things. At the same time, it is obvious that the assemblage would be precisely the same as though the unit thing had been repeated straight away to x times, and the things thus obtained then arranged in a square group. Hence $x=n^2$.

But we saw, from Figs. 33 and 34, that $8=\sqrt[2]{4^3}$; and it follows that, in a similar manner, any particular number may be obtained by raising any other number to a suitable integral power, and then extracting some integral root of the resulting assemblage—which, according to the "law" indicated by the symbolism we are seeking to justify, is

the same thing as raising the selected number to the corresponding fractional power.

On this basis, if we say that n must be obtainable by raising x to some suitable power (say p), the statement might be written

$$\sqrt[2]{x^1}=n=x^p\,;$$

or, dispensing with n now that it has so far served its purpose—

$$\sqrt[2]{x^1}=x^p.$$

The number of things in the row is, of course, unchanged. We have simply *called* it x^p, instead of *calling* it n.

If now each thing in the row be repeated to x^p times to form a layer, as before, the assemblage will comprise $(x^p)\times(x^p)=x^{p+p}=x^{2p}$ things (see p. 68); which, of course, is the same number as x^1.

Therefore, since $x^1=x^{2p}$, it follows that $2p=1$; so that $p=\frac{1}{2}$; giving

$$\sqrt[2]{x^1}=x^{\frac{1}{2}}.$$

Similarly, it may be shown that

$$\sqrt[3]{x^1}=x^{\frac{1}{3}}\,;\quad \sqrt[4]{x^1}=x^{\frac{1}{4}}\,;\ \text{and so on.}$$

Proceeding on the same basis of argument, we may justify the general statement $\sqrt[q]{x^p}=x^{\frac{p}{q}}$; which, for our purposes here, may be more conveniently expressed as $x^{p(\frac{1}{q})}$.

Taking, as an example, $\sqrt[7]{x^4}$, we know that x^4 denotes an assemblage of things; and these things may be arranged in a single row. If the number of things in this assemblage be k, their arrangement in a single row may be recorded as

$$x^4=k^1\,;$$

and writing k^1 in place of x^4, the symbol becomes $\sqrt[7]{k^1}$.

Now, it has been shown that this may be expressed as $(k)^{\frac{1}{7}}$; and replacing k by its equivalent number x^4:

$$\sqrt[7]{x^4}=(x^4)^{\frac{1}{7}}=x^{\frac{4}{7}}\,;$$

from which the general expression $\sqrt[q]{x^p}=x^{\frac{p}{q}}$ follows naturally.

Another view of $x^{\frac{1}{2}}$, with which the author has been able to assist some students to a clearer understanding, may be stated as follows: Regard $x^{\frac{1}{2}}$ as an abbreviated form of the equivalent symbol $x^{(1\div 2)}$; and interpret this as meaning one unit thing repeated to x times to form a row (denoted by x^1), and then the things comprised in this row *arranged to form an Involution group of the Second Order* (denoted by $\div 2$).

Similarly, for $x^{\frac{1}{3}}$, suppose the unit thing repeated to x times to form a row, and the things comprised in this row arranged to form an Involution group of the Third Order.

For $x^{\frac{4}{7}}$, suppose the unit thing repeated to x^4 times; the resulting assemblage first arranged in a single row, and then rearranged to form an Involution group of the Seventh Order. And so on for any other case of fractional indices.

It cannot be too carefully borne in mind that the sole advantage to be gained by the use of fractional indices, from the practical point of view, lies in their consistency with the basis of logarithms. In setting out ordinary calculations dealing with real things, if the physical operation to be performed upon an assemblage of things were that which is properly represented by the symbol, *e.g.*, $\sqrt[4]{x^5}$, it would be preferable from all points of view that the statement be recorded in that way. Then, if the number represented by the symbol is to be determined by the use of logarithms (as would be necessary in a case such as that chosen by way of an example), let the symbol be written as *equivalent to* $x^{\frac{5}{4}}=x^{1\cdot 25}$. Such small sacrifices to the god of method cost very little in either time or trouble; while they serve the extremely useful purpose of keeping one's own vision clear, and of helping to prevent the introduction of unnecessary and avoidable vagueness for those who come after.

Negative fractional indices do not call for detailed explanation; for it follows, from the foregoing discussion, that (*e.g.*) $x^{-\frac{2}{3}}$ denotes merely that one unit thing is to be divided into $\sqrt[3]{x^2}$ equal parts, and that the same operation might be signified by the symbol $\left(\frac{1}{\sqrt[3]{x^2}}\right)$.

Logarithms.—As has been stated already, logarithms are merely indices; and their applicability to practical calculation is based on the fact (demonstrated in Chapter IV.) that any particular number may be obtained by raising any other chosen number to a suitable power.

The number chosen for raising to a power is called the "base"; and the number which indicates the power to which that base must be raised is called the "logarithm" of the particular number required, with regard to the chosen base.

There is no need for discussion here, either as to the significance of the words "mantissa" and "characteristic," which are used to denote the fractional and integral parts (respectively) of logarithms, or as to the method by which tables of "common" logarithms (*i.e.* logarithms which have 10 for their "base") are arranged in order to make the tables at once concise, convenient, and comprehensive for practical purposes. These matters are well described and explained in many admirable text-books; and they usually present little difficulty to students.

At the same time, however, there are a few points regarding the elementary principles of logarithms which should be understood more clearly than is common, if true facility and confidence in their manipulation is desired—and which are, moreover, of considerable importance because of their bearing upon other branches of the work.

It is easy to understand such simple cases as:

$$\text{since } 10^2 = 100, \text{ therefore Log } 100 = 2;$$

but there is often some difficulty in obtaining a proper

appreciation of the physical significance attaching to such cases as :

$$\text{Log } 2 = 0{\cdot}30103\text{ ; and Log } 3 = 0{\cdot}4771213.$$

Of course, there is nothing difficult in applying the "rule" to express the statements in the form

$$10^{0{\cdot}30103} = 2\text{ ; and } 10^{0{\cdot}4771213} = 3\text{ ;}$$

but to most students there is also nothing very illuminating or really helpful in so doing.

A better way of approaching the matter is to remember that a fractional index is really a combination of an integral power and an integral root (see p. 76) ; and, for this reason, to express such a logarithm in the form of a vulgar fraction when it is written definitely as an "index"—thus

$$10^{\left(\frac{30103}{100000}\right)} = 2\text{ ; and } 10^{\left(\frac{4771213}{10000000}\right)} = 3.$$

From this it is clear that the same statements might be made more precisely in the form

$$2 = \sqrt[100000]{10^{30103}}\text{ ; and } 3 = \sqrt[10000000]{10^{4771213}}\text{ ;}$$

and from this, in turn, it follows that if we would obtain 2 by operating upon 10 with the processes of Involution and Evolution, we have but to first raise 10 to the 30103-rd power, and then to take the 100,000th root of the result.

In other words, we could obtain a row of 2 things by first repeating an original unit thing to 10 times to form a row, repeating each thing in the row to 10 times to form a layer, repeating each thing in the layer to 10 times to form a block, repeating each thing in the block to 10 times to form a row of blocks, and so on until 10 had occurred as a factor of repetition 30103 times ; and then rearranging the things comprised in the resultant assemblage so that they form an Involution group of the 100,000th order.

Conversely, since it is obvious that the number of things in the total assemblage so obtained is not altered by mere rearrangement :

$$10^{30103} = 2^{100000}\text{ ; or } 2^{100000} = 10^{30103}\text{ ;}$$

and if the things forming the assemblage be rearranged in an Involution group of the 30103rd order,

$$\sqrt[30103]{2^{100000}} = 10, \text{ or } 2^{\left(\frac{100000}{30103}\right)} = 10\ ;$$

from which it follows that a row of 10 things may be obtained by first repeating an original unit thing to 2 times, repeating such repetition to 100,000 times, and then rearranging the things comprised in the resultant assemblage so that they form an Involution group of the 30103rd order.

The task of deducing an interpretation on similar lines for log $3 = 0{\cdot}4771213$ and other typical cases may be left as an exercise for the interested student.

Now, it is obvious, from this interpretation, that logarithms are in no way mysterious or difficult to understand, and that they are but symbols indicating the arrangement and counting of real things in groups, just as are all other numbers which may be properly used in practical calculation. Even the actual determination of $\log_{10} 3$, by physical means alone, and to any reasonable degree of approximation, is seen to be not only possible, but easily within the power of any normal human being possessed of the necessary leisure and diligence. And while the demonstration of such a case as log $2 = 0{\cdot}30103$, by actually arranging and rearranging real things, would be a somewhat lengthy and tedious business, there is nothing intrinsically difficult about it. It would not call for any higher human qualities than care and patience—though it would certainly demand these in generous measure; and satisfactory performance of the operation would provide unquestionable evidence of mental and physical endurance, but none at all of superhuman intelligence.

With other—and far less troublesome—methods available by which logarithms may be determined, there is no need for such feats; but that fact does not in any way detract from the reality of the processes on which such methods are based.

Positive Characteristics.—As might be expected, an equally simple interpretation may be applied to logarithms having positive characteristics.

For instance, in the case of log $300 = 2{\cdot}4771213$ it is only necessary to rewrite the statement in the forms

$$10^{2\cdot4771213} = 10^{2+0\cdot4771213} = 10^2 \times 10^{\left(\frac{4771213}{10000000}\right)}$$
$$= 100 \times \left\{\sqrt[10000000]{10^{4771213}}\right\};$$

when the physical operations of repetition, arrangement and rearrangement of real things which are implied become plainly apparent.

The reasons for the addition, subtraction, multiplication and division of logarithms, to effect the multiplication, division, involution and evolution (respectively) of the numbers to which they relate, need no explanation beyond that which follows obviously from the foregoing discussion, so long as the negative characteristic is not introduced.

It will be obvious that nearly all logarithms are approximate only. For instance, the common logarithms of 2 and 3 as quoted above are correct to the seventh decimal place; but logarithms have been determined and tabulated to a much higher degree of accuracy for use in such calculations as those concerned with Astronomy, where the numbers to be dealt with are extremely large.

For most purposes of practical engineering, four-figure logarithms—*i.e.* mantissae correct to the fourth decimal place—are sufficient; and since a complete table on this basis may be printed with great clearness on two pages of little more than pocket-size paper, such tables are very widely used. Occasionally one finds cases in which errors have been introduced through the use of four-figure logarithms when a more precise determination should have been employed; and care is necessary in this respect. Excellent tables are available, giving logarithms to five, six and seven decimal places; and there are few purposes in ordinary engineering work for which the tables available,

used with reasonable care and discernment, are not sufficient.

Since there are many more numbers between 10 and 100 than there are between 1 and 10, it follows that the differences between the logarithms of successive numbers in the latter range must inevitably be greater than the differences between the logarithms of successive numbers in the former range. It is by reason of this fact that the fourth-figure differences in some tables of four-figure logarithms vary by such wide steps in the first few lines of the tables ; and the consequence is that these fourth-figure differences are not really constant over the whole range (from 0 to 9) of the third figures in the mantissae to which they apparently relate. During the last few years this defect has been remedied, to a considerable extent, by " stepping " the first ten or twenty lines of the four-figure tables, giving two sets of fourth-figure differences for each line—one applicable to the mantissae of logarithms relating to numbers in which the third figure is 4 or less, and the other for those relating to numbers in which the third figure is 5 or more.

Of course, in a great deal of practical calculation, results correct to two—or, at most, to three—decimal places are sufficient ; but where a higher degree of accuracy is required it is well to use more comprehensive tables than the ordinary four-figure logarithms if practicable.

Negative Characteristics.—Numbers between 0 and 1, being themselves fractions, are necessarily represented by negative powers of 10.

The reason for this will be apparent on consideration of the fact that, since a number between 0 and 1 indicates a fraction (or " part ") of one unit thing, even were 2 chosen as the " base," the index would be negative, in order to denote the operation of dividing into some number of equal parts, as opposed to the process of ordinary repetition, denoted by a positive index. Much more so, then, since the " base " is 10.

In view of the inferences drawn, in Chapter IV., from the

discussion on Involution, there is no need for detailed explanation here of the fact that the logarithm of 1 *to any base* is 0; but it will be noticed that this fact clearly marks the passage from logarithms which are wholly positive to those which are wholly negative.

Were it not for the device of combining negative characteristics with positive mantissae, tables of logarithms would, of course, be rendered much more complicated and voluminous than they are; for, apart from the difference of sign, it would be necessary to tabulate the logarithms of numbers more than 1 separately from those of numbers less than 1, since the mantissa would not be the same in the two groups for numbers comprising the same significant figures in the same order—*e.g.* while $\log 3 = 0{\cdot}4771213$, $\log 0{\cdot}3 = -0{\cdot}5228787$.

It should be carefully borne in mind always that the logarithms of numbers between 0 and 1 are really negative throughout, and not a mixture of positive and negative parts. They are used in the latter form solely for the purpose of simplifying the tables of logarithms. Many errors in calculation, and—which is even worse—many false impressions regarding the operations performed, have been caused by failure to realise this and to keep it clearly in view.

The logical significance of such a statement as

$$\text{Log } 0{\cdot}2 = \bar{1}{\cdot}30103$$

is quite plain, and may be traced thus:

$$0{\cdot}2 = \frac{1}{\left(\sqrt[100000]{10^{69897}}\right)} = \frac{1}{\left(\sqrt[100000]{10^{69897}}\right)} \times \frac{\left(\sqrt[100000]{10^{30103}}\right)}{\left(\sqrt[100000]{10^{30103}}\right)}$$

$$= \frac{\left(\sqrt[100000]{10^{30103}}\right)}{\left(\sqrt[100000]{10^{100000}}\right)} = \frac{\left(\sqrt[100000]{10^{30103}}\right)}{10^{1}}$$

$$= 10^{(0\cdot30103 - 1\cdot00000)}.$$

A method of regarding these "mixed" logarithms which the author has found very useful is to think of them in

much the same way as that in which a cashier thinks of his "petty cash" account. Since he must always have money in hand to meet small demands, he must draw cash to keep his account floating; but instead of drawing advances in all sorts of odd and unrelated amounts, it is found preferable for him to draw always in a certain specified round sum—or in multiples of that sum—at a time.

The positive mantissa of a "mixed" logarithm may well be regarded as the "balance in hand" of a petty cash account; and the negative characteristic as the cash drawn—recorded as a debt or liability until completely accounted for, but always partly offset by the balance in hand.

In the case of log 0·2 as above, for instance, a disbursement of 0·69897 may be supposed to have become necessary; but we do not draw an advance of exactly that amount, for it is more convenient to draw in either the specified round sum or in multiples of it. Here we draw the specified round sum—*i.e.* 1—and after paying for the purchase in question the state of the petty cash account might be recorded as:

Assets—balance in hand = +0·30103.
Liabilities—advance drawn = −1·00000.

In such a case as log 0·00003, the disbursement requiring to be made would amount to 4·5228787; and here we should draw an advance of five times the specified round sum—the smallest multiple sufficient to cover the outlay—recording the state of the petty cash account, after paying for the purchase, as:

Assets—balance in hand = +0·4771213.
Liabilities—advance drawn = −5·0000000.

As will be seen presently, this method is likely to be found very helpful in manipulating "mixed" logarithms in practical calculations, where such logarithms must be added together, one subtracted from another, and multiplied or divided by some number for the purpose of raising to a power or extracting a root.

Uniform Rule for all Characteristics.—As is well known,

it is usually taught that there is one rule for the determination of positive characteristics, and a different rule for negative characteristics.

These two rules being at once well known to students and explained fully in large numbers of books, there is no need to discuss them here ; but it has always been a matter of some difficulty to the author to understand why *two* rules should have been devised—with the inevitable tendency to confusion and risk of error, just where everything is wanted to be most clear and safe—when the facts indicate so plainly that one uniform rule will cover the whole range completely. Moreover, this single uniform rule is much more simple and straightforward to apply than even the simpler of the two so generally taught—as is only to be expected, for it is just a natural statement or record of the facts from first-hand observation.

Let us consider, as a typical instance covering a fairly indicative range, log 0·003, log 0·3, log 30·0 and log 3000·0.

In each case the positive mantissa is 0·4771213, and by the orthodox "rules" the characteristics would be $\bar{3}$, $\bar{1}$, 1 and 3.

Now, considering the facts of logarithms as they stand, we see that

$$
\begin{array}{llll}
\log 10000 & = \log (10^4) & = +4 ; & \\
\log 1000 & = \log (10^3) & = +3 ; & \\
\log 100 & = \log (10^2) & = +2 ; & \\
\log 10 & = \log (10^1) & = +1 ; & \\
\log 1 & = \log (10^0) & = 0 ; & \\
\log 0{\cdot}1 & = \log (\frac{1}{10}) & = \log (10^{-1}) & = -1 ; \\
\log 0{\cdot}01 & = \log (\frac{1}{100}) & = \log (10^{-2}) & = -2 ; \\
\log 0{\cdot}001 & = \log (\frac{1}{1000}) & = \log (10^{-3}) & = -3 ; \\
\text{and } \log 0{\cdot}0001 & = \log (\frac{1}{10000}) & = \log (10^{-4}) & = -4.
\end{array}
$$

This range may, of course, be extended indefinitely in both directions according to the order of procedure clearly established ; but sufficient is shown for all the purposes of illustration required here.

A little consideration will show that all numbers lying between *one thousand* and *ten thousand* will have common logarithms between 3 and 4—*i.e.* their characteristics will be $+3$ in all cases; and clearly, all such numbers may be classified and thought of as "*numbers of thousands.*"

Turning to the interval of the range comprising numbers between 0·001 and 0·01, it is obvious that all numbers in this range will have logarithms between -3 and -2 —*i.e.* with a positive mantissa, the characteristic throughout this range must be $\bar{3}$; and since the numbers marking the limits of the range are *one thousandth* and *ten thousandths* (*i.e.* one hundredth), it follows that all numbers in the range may be classified and thought of as some "*numbers of thousandths.*"

Thus we see that all numbers which are "*numbers of thousands,*" and also all numbers which are "*numbers of thousandths,*" have 3 for the characteristic of their common logarithms—positive in the former, and negative in the latter class. And so the "uniform rule" begins to emerge.

Similarly it will be seen that all numbers between 100 and 1000 (which may all be regarded as "*numbers of hundreds*") have $+2$ for the characteristic of their common logarithms; while all numbers between 0·01 and 0·1 (which, since they lie between $\frac{1}{100}$ and $\frac{10}{100}$, may all be regarded as "*numbers of hundredths*") have $\bar{2}$ for the characteristics of their common logarithms.

All numbers between 10 and 100 may be regarded as "*numbers of tens,*" and will have $+1$ for their characteristics; while all numbers between 0·1 and 1·0 may be regarded as "*numbers of tenths,*" and will have $\bar{1}$ for their characteristics.

This leaves only a single range—*i.e.* numbers between 1 and 10—without a counterpart; and as all such numbers may be regarded as "*numbers of ones,*" having 0 for the characteristics of their common logarithms, this range forms an excellent "centre-line," as it were, about which

all other numbers are grouped in regular and symmetrical ranges.

This "uniform rule for all characteristics" may, then, be stated in some such forms as this:

(1) Characteristics are positive for all numbers greater than 1; and negative for all numbers less than 1.

(2) As regards magnitude, characteristics are:

for "*numbers of ones,*" 0;
for "*numbers of tens*" and "*numbers of tenths,*" 1;
for "*numbers of hundreds*" and "*numbers of hundredths,*" 2;
for "*numbers of thousands*" and "*numbers of thousandths,*" 3; and so on.

One of the greatest advantages of the "rule," however, lies in the fact that it need not be formulated as a rule at all, since it is neither more nor less than a plain statement of the actual facts. The author has found it of real service in practical use, not only because it does away with the need for remembering and applying two "rules" which are at once sufficiently similar and sufficiently dissimilar to be troublesome and irritating, but also—and, perhaps, more—because it reduces the liability to error by focussing the attention upon the actual facts of the case in hand.

It is commended to the practical student for consideration, with a view to his according it a fair trial in practical work.

Logarithms of Negative Numbers.—It is demonstrated in Chapter II. that the question of "positive nature" as opposed to "negative nature" arises only when we are considering the effects (upon groups of things) of things which pass—or are required to pass—into or out of those groups; and it is hoped that the demonstration has been —after subjection to the most searching and critical investigation—accepted by the student as a simple exposition of very plain facts.

It is, however, somewhat curious to find that failure on the part of leading teachers and writers to observe this fact has led to much fallacious doctrine in mathematics.

Some instances of this we have pointed out already, and others are referred to in later chapters; while at the moment we are concerned with one—viz., the belief that "*a negative number has no logarithm*"—which has quite recently done a good deal of harm in wasting time and effort when both could ill be spared.

As regards principle, of course, the fallacy is the same as that which led to the statements, "The square of a negative quantity is positive; the cube of a negative quantity is negative; and so on" (see p. 90); and "It is not possible to extract the square root of a negative quantity and obtain a real result" (see p. 90); and the inferences to which it leads are no less erroneous here than there.

The physical operations concerned in the determination of the logarithm corresponding to a certain number with regard to a particular base have been shown (see p. 109) to be mere arrangements and rearrangements of real things; and if the determination be effected through logical argument and calculation instead of by physical means, it is none the less imperative that the argument and calculation, besides agreeing with the conclusions, shall follow precisely the course of the appropriate physical processes. As a fact, it may be shown without difficulty that the methods of reasoning by which logarithms are determined for tabulation *do* so represent physical operations.

A number and its logarithm are, therefore, related only on the basis of the arrangement and counting of things, and not in any way with respect to the nature of those things or to their effects and influences upon extraneous groups of things.

We use and manipulate the logarithms of *numbers*, and not of *natures*; the latter remaining unaffected by the processes applied.

If it be required to determine or estimate the number of things comprised in a group indicated by such an expression as

$$\mathrm{N} = \frac{(8\cdot7)^3 \times \sqrt{47\cdot2} \times (-8\cdot1)}{(51\cdot73)^2 \times \sqrt[3]{849\cdot7}},$$

it is only necessary to notice, from the recorded facts, that the things are of such nature that they would have the effect of reducing the assets of any other group of similar things to which they might be added; and the number N may then be calculated in the ordinary way.

For the sake of clearness in record, and to the avoidance of doubtfulness, the steps should be recorded thus:

$$N = -\left\{\frac{(8\cdot7)^3 \times (47\cdot2)^{\frac{1}{2}} \times (8\cdot1)^1}{(51\cdot73)^2 \times (849\cdot7)^{\frac{1}{3}}}\right\} = -1\cdot475.$$

Manipulation of "Mixed" Logarithms.—In manipulating "mixed" logarithms for the purposes of practical calculations, the device of the "petty cash account," explained in the discussion on "Negative Characteristics" (p. 114), will be found useful.

As an easy example, suppose that we have to evaluate

$$(8694 \times 0\cdot09671 \times 0\cdot000623 \times 0\cdot5736).$$

Taking logarithms, and setting them out for addition in the usual way:

$$\begin{array}{lll}
(a) \ \log 8694\cdot0 & & = 3\cdot9392 \\
(b) \ \log & 0\cdot09671 & = \bar{2}\cdot9854 \\
(c) \ \log & 0\cdot000623 & = \bar{4}\cdot7945 \\
(d) \ \log & 0\cdot5736 & = \bar{1}\cdot7587
\end{array}$$

These four statements may be regarded as four "daily records" of the petty cash account—*i.e.* each a statement recorded at the close of a day, showing the "*balance in hand at close*" and the "*total advances drawn during the day*"; and the process of adding them may be regarded as corresponding to that of summarising the four separate records into a single statement covering the four days' working of the account.

The four records might be more clearly stated in tabular form, thus:

Day.	Assets. (Balance in hand.)	Liabilities. (Advance drawn.)	Disbursements.
(*a*) . .	3·9392	0	0
(*b*) . .	0·9854	2	1·0146
(*c*) . .	0·7945	4	3·2055
(*d*) . .	0·7587	1	0·2413
Totals. .	6·4778	7	4·4614

Now, as there is a considerable balance in hand, it might be thought well to *repay* advances drawn, so far as possible. If the whole balance (6·4778) were handed over, the net result would be :

$$+6{\cdot}4778-7{\cdot}0=-0{\cdot}5222\,;$$

indicating a deficit of 0·5222.

The total disbursements, added to the total balance in hand ready for handing over, amount to $4{\cdot}4614+6{\cdot}4778=10{\cdot}9392$; and this, of course, is equal to the sum of the total advances drawn and the initial balance, $7+3{\cdot}9392=10{\cdot}9392$, showing that the account is correct as regards the agreement of records.

Unless the account were being definitely and finally closed, however, it would be preferable to *repay* advances from the balance in hand by multiples of the specified round sum in which the advances are *drawn*; and thus, it would be better to retain the "petty" balance—*i.e.* the *portion* of a single unit advance—recording also the outstanding advance of 1 which remains after the 6 have been repaid from the balance to offset the total advance of 7, in readiness for the succeeding day's working of the account.

The state of the account might then be recorded as

$$+0{\cdot}4778-1{\cdot}0\,;$$

or, more conveniently (for our purposes), as

$$\bar{1}{\cdot}4778.$$

On reference to the tables of antilogarithms, $\bar{1}{\cdot}4778$

is found to be the logarithm of 0·3004 ; and this is, therefore, the evaluation sought.

Several highly interesting and instructive inferences may be drawn from such discussion as the foregoing; but they will probably be readily apparent to the student on consideration of similar cases.

One such inference is, however, worthy of notice in passing. Regarding the three fractions in the original expression as operating upon the first term (8694), this number might be written, in the nomenclature of Involution, as $\{10^{3\cdot9392}\}$ things. The total "disbursements" amounting to 4·4614, the effect of the fractions operating upon the first term may be recorded as

$$\{10^{(3\cdot9392-4\cdot4614)}\}\text{ things}=\{10^{(-0\cdot5222)}\}\text{ things}$$

$$=(\text{approx.})\left\{\frac{1}{10^{0\cdot5}}\right\}\text{ things}$$

$$=(\text{approx.})\ \frac{1}{\sqrt{10}}\text{ things ; or (approx.) }\frac{1}{3\cdot3}\text{ things ;}$$

thus keeping the relation to real things clearly in view throughout the calculation.

Had the first term in the given expression been a fraction as well as the others, instead of a fairly large whole number (8694), the first logarithm would have been "mixed" instead of wholly positive. Consequently the "petty cash account" would have shown some "advance drawn" for the first day, and also some "disbursements," instead of none. The principle remains unchanged, of course; and the student is advised to satisfy himself completely on this point by fully working a few typical examples on the lines indicated, so that all aspects of the matter may be seen with the processes in operation.

A second example may now be considered, introducing the subtraction of "mixed" logarithms.

Suppose it be desired to evaluate

$$\frac{7\cdot462\times0\cdot0472\times0\cdot7854}{23\cdot27\times0\cdot0364}.$$

Taking logarithms, the first step may be set out thus :

Logarithms of Numerator.	*Logarithms of Denominator.*
Log 7·462 $= 0·8728$	Log 23·27 $= 1·3668$
Log 0·0472 $= \bar{2}·6739$	Log 0·0364 $= \bar{2}·5611$
Log 0·7854 $= \bar{1}·8951$	
Total $= \bar{1}·4418$	Total $= \bar{1}·9279$

Now, this state of affairs may be regarded as the equivalent of a petty cash account which stands at :

Balance in hand $= +0·4418$; *Advance drawn* $= -1·0$;

and which is then called upon to make a disbursement of 0·9279, while a discount of 1·0 in respect of some previous transaction is returned to it.

Had the balance in hand (0·4418) been sufficient to meet the disbursement called for, the latter could have been effected straight away, leaving only the recovered discount to be adjusted.

As it is *not* sufficient, a further advance of 1 must be drawn, making the balance in hand 1·4418, the total advance drawn 2, and the state of the account $+1·4418-2·0$. If now the disbursement of 0·9279 be made, the balance in hand will be reduced to $+1·4418-0·9279=+0·5139$; and the state of the account will then be $+0·5139-2·0$. The discount of 1 recovered may be handed over to the source from which advances are drawn, as repayment of an advance, reducing the liabilities of the petty cash account to 1 ; and the final state of the account may be recorded as $+0·5139-1·0$, or $\bar{1}·5139$.

On reference to the tables of antilogarithms, it is found that $\bar{1}·5139$ is the logarithm of 0·3265 ; which, therefore, is the evaluation sought.

The student should argue out for himself, on the lines indicated above, cases which lead to such statements as ·

(i.) Log of Numerator $= \bar{2}·8471$; Log of Denominator $= 3·4267$;
(ii.) Log of Numerator $= \bar{4}·2063$; Log of Denominator $= \bar{6}·9517$;
(iii.) Log of Numerator $= 3·1278$; Log of Denominator $= \bar{2}·4263$;

and others to introduce the conditions likely to arise in practical calculations.

One more example, indicating the manner in which "mixed" logarithms may be manipulated in calculations dealing with Involution and Evolution, will suffice.

Suppose it be desired to evaluate

(*a*) . . . $(0{\cdot}6387)^4$;
(*b*) . . . $\sqrt[3]{0{\cdot}0009381}$; and
(*c*) . . . $\sqrt[2]{(0{\cdot}7284)^3}$.

For the first portion (*a*) of the example :

$$\log 0{\cdot}6387 = \bar{1}{\cdot}8053; \text{ and therefore } \log (0{\cdot}6387)^4 = 4 \times \bar{1}{\cdot}8053.$$

Now, this may be regarded as the equivalent of summarising four "daily records" of the "petty cash account," the statement made at the close of each of the four days being :

Balance in hand $= 0{\cdot}8053$; *Advance drawn* $= 1$.

The summary of the four days' working, therefore, would show :

$$\textit{Total balance in hand} = 4 \times 0{\cdot}8053 = 3{\cdot}2212.$$
$$\textit{Total advances drawn} = 4 \times 1 = 4.$$

On the lines already indicated in previous examples, the 3 whole "*advance units*" in the "balance in hand" may be repaid to the source from which advances are regularly drawn, offsetting 3 of the 4 unit advances drawn, and leaving 1 outstanding.

The state of the account might then be recorded as

$$+0{\cdot}2212 - 1{\cdot}0, \text{ or } \bar{1}{\cdot}2212.$$

On reference to the tables of antilogarithms, it is found that $\bar{1}{\cdot}2212$ is the logarithm of $0{\cdot}1627$; and hence we conclude

$$(0{\cdot}6387)^4 = 0{\cdot}1627.$$

For the second portion (*b*) of the example :

$$\log 0{\cdot}0009381 = \bar{4}{\cdot}9722\text{ ; and } \log \{\sqrt[3]{0{\cdot}0009381}\}$$
$$= \tfrac{1}{3} \times \bar{4}{\cdot}9722.$$

Here we may imagine a " petty cash account " standing at :

Balance in hand $= 0{\cdot}9722$; *Advances drawn* $= 4$;

and we are asked to close this single account, and to open in its place three separate divisional accounts—as might easily occur in business—subject to the condition that each new account must comprise a *balance in hand* and some record of *unit advances drawn*—as they should, to be properly constituted for working purposes.

For clearness, it is well to note that the real state of the account before the division is represented by a deficit of $4 - 0{\cdot}9722 = 3{\cdot}0278$; and this, of course, must not be either increased or diminished by the division—*i.e.* no " disbursements " are permitted until after the division shall have been effected.

The operation may be satisfactorily performed by drawing two further *unit advances* (so that the total advances drawn shall be the lowest practicable multiple of the number of separate accounts to be opened—in this case, 3), and placing these with the balance in hand for division.

This will give the state of the account as :

Balance in hand $= 2{\cdot}9722$; *Advances drawn* $= 6$;

and dividing this into three equal parts, the state of each new account will be :

Balance in hand $= 2{\cdot}9722 \div 3 = 0{\cdot}9907\dot{3}$;
Advances drawn $= 6 \div 3 = 2$;

which may be stated as $0{\cdot}9907\dot{3} - 2{\cdot}0$, or $\bar{2}{\cdot}9907\dot{3}$.

Each new account will therefore stand at a deficit of $1{\cdot}0092\dot{6}$; and the three accounts together will represent a total deficit of $3 \times 1{\cdot}0092\dot{6} = 3{\cdot}0278$, as did the original single account.

On reference to the tables of antilogarithms, it is found that $\bar{2}\cdot 9907\dot{3}$ is the logarithm of $0\cdot 09789$; and hence we conclude

$$\sqrt[3]{0\cdot 0009381}=0\cdot 09789.$$

For the third portion (*c*) of the example:

log $0\cdot 7284=\bar{1}\cdot 8623$; and therefore log $\{\sqrt[2]{(0\cdot 7284)^3}\}$ $=\frac{3}{2}\times\bar{1}\cdot 8623$.

This is a combination of the two previous cases; for we have, clearly, to summarise three exactly similar "daily records" of the "petty cash account," close the single account, and open two separate divisional accounts in its place.

Further description of the method is unnecessary, and the processes may be merely recorded thus:

$$\bar{1}\cdot 8623\times 3=(+0\cdot 8623-1\cdot 0)\times 3=+2\cdot 5869-3\cdot 0$$
$$=+1\cdot 5869-2\cdot 0$$
$$(+1\cdot 5869-2\cdot 0)\div 2=+0\cdot 7935-1\cdot 0=\bar{1}\cdot 7935.$$

On reference to the tables of antilogarithms, it is found that $\bar{1}\cdot 7935$ is the logarithm of $0\cdot 6215$; and hence we conclude

$$\sqrt[2]{(0\cdot 7284)^3}=0\cdot 6215.$$

Needless to say, it is not suggested that the student should wade through the whole argument each time he has to manipulate a few "mixed" logarithms. It will, however, be well worth his while to do so with a fair number of good examples typical of—and, preferably, taken from—actual practice. By this means he will soon find himself possessed of a method which is at once far quicker and less troublesome than those usually employed; he will acquire the faculty of keeping the realities involved plainly in view, without effort, throughout calculations, rendering errors both less likely to occur, and likely to be less serious in the event of their occurring; and last—though by no means least—he will acquire something of that "power of perception" by which he will be enabled to perform whole processes mentally with precision, and to, as it were,

sense the required result with sufficient accuracy for all ordinary practical purposes.

It has been splendidly said that KNOWLEDGE is PERCEPTION; and—at least in the practical utilisation and control of mathematics for engineering purposes—it appears equally incontrovertible that Perception is Knowledge.

One word of advice is offered here to the student who may be in need of such for his future welfare and guidance: logarithms are (like so many other human devices) excellent servants, but they are extremely bad masters. Let the student be careful to remain the employer of logarithms—not their slave; and let him use them only when their use is either necessary or desirable in the interests of facility and expedition. The ordinary arithmetical processes, suitably abbreviated by the omission of all unnecessary mechanical work, and intelligently applied, are both quicker and easier than logarithms for many classes of work; while they are far less likely to introduce those silly mistakes which are so easily made when carrying out some complicated piece of work by means of logarithms—mistakes which are nearly always due less to a slip in applying some tricky "rule" than to the fact that the basic realities have been completely lost sight of.

The Slide Rule.—In principle, the ordinary slide rule is nothing more than a device by which logarithms may be added and subtracted by *lengths* instead of by figures. The *logarithms* are set down to a convenient scale, starting all from the left-hand end—in exactly the same way as are distances on an ordinary two-foot rule; and against the mark which shows the right-hand end of a scaled representation of a *logarithm* is written the *number* to which that logarithm corresponds.

The determination of squares and square roots is provided for by making the ordinary rule to read against another on which the logarithms are represented to half the scale of the ordinary rule—the logarithm of a number on the new

rule occupying only one-half the length allotted to the same logarithm on the first rule. Similar devices for making rules to yield cubes and cube roots directly are sometimes used ; and the student will do well to analyse such devices for himself by reference to the actual slide rules in which they are embodied. Another extension is the " log-log " scale, which is made to show the *logarithms-of-logarithms-of-numbers* ; and there are many more—mostly devised to meet particular needs in special branches of calculation. As to whether such calculations are performed in practical work with such frequency as to call for the possession of a special slide rule is, however, open to some question.

A clearly-marked, reliable and easy-working slide rule is an extremely handy thing for an engineer to have by him ; but he should be careful to prevent enthusiastic admiration for its ingenuity from leading him to use it when other methods are more suitable and effective. The author prefers, for his own work, an ordinary 10 in. " Standard Pattern " rule of good quality—a homely instrument, with no pretensions to magical powers. Even a slide rule should be used in conjunction with, and not in place of, an ordinary, normal brain.

While the slide rule serves many purposes, however—and serves them well if properly used—it is not capable of performing all the calculations of engineering alone and unaided. No matter how expensive and awe-inspiring, a slide rule is but a poor substitute—as an item of equipment for the practical engineer—for a decent table of logarithms. Rule and table have each their own particular fields of utility ; and neither can be made to compete successfully with the other in its own particular field.

The student will do well to acquire facility and dexterity in the use and proper control of *all* aids to calculation—all, that is, which will help to relieve him of merely mechanical work, and thus leave his brain free for the exercise of its proper functions of perceiving, interpreting, originating and directing. But if he is wise, he will not hesitate for

one moment to abandon and discard any implement, device or method which he finds in the slightest degree tending to obstruct or obscure his intimate view and knowledge of the actual realities with which he is concerned.

Napierian Logarithms.—So far as is known, Baron Napier was the inventor of logarithms as a system—*i.e.* he was the first to devise a table showing the logarithms of numbers, all with respect to one "base"; and it is for this reason that the logarithms of his system are known as *Napierian* logarithms. They are also spoken of sometimes as *Hyperbolic*, and sometimes as *Natural* logarithms.

Though the work of actually calculating the first table of Napierian logarithms appears to have been executed by Briggs (who is generally credited with having subsequently invented "common"—or Briggian—logarithms), the designation of the Napierian system by the name of the man who introduced it is at least reasonable; and it is preferable that such designation be employed, rather than the other two referred to above.

The term "hyperbolic" was applied to the logarithms of the Napierian system, because it was found that these logarithms were connected by relations which were known to govern the asymptotic spaces of the hyperbola; and it was apparently thought that this property was derived by the logarithms, in some way, from the hyperbola. As a fact, the logarithms of any system would follow the same relation in principle, the only variation being that the angle between the asymptotes would vary with the "base" of the system. The hyperbola which corresponds with the Napierian system has its asymptotes mutually perpendicular; whereas the angle between the asymptotes to correspond with the common system is $25°\text{-}44'\text{-}25''$ (nearly). Since this fact was demonstrated, the use of the term *hyperbolic* has been much less generally applied to Napierian logarithms than formerly.

The "base" of the Napierian system is $2{\cdot}7182818\ldots$, an incommensurate quantity. of which the value stated

is correct to the seventh decimal place. This is usually denoted by the symbol " ϵ "; and it may be evaluated to any desired degree of approximation from the relation: $\epsilon = (1 + \frac{1}{n})^n$, n being infinitely great. Expansion of this by the Binomial Theorem gives:

$$\epsilon = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots .$$
$$= 1 + 1 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \ldots .$$

It was because Napierian logarithms arose from this apparently " natural " series that they were called *natural* logarithms. Such a title has, however, no more true applicability to this than to any other system of logarithms; and it is now seldom used.

Napierian logarithms, though necessary for a few special purposes, are very little used in practical calculation as compared with common logarithms; though in constructing a table of common logarithms it is necessary to calculate at least some logarithms on the Napierian system, translating them to the common system by multiplying them by a "modulus"—viz. $\frac{1}{\log_{\epsilon} 10} = \frac{1}{2 \cdot 3025851} = 0 \cdot 4342945.$

Sufficient reason for the displacement, in practical calculation, of the Napierian by the common system of logarithms will be plainly seen from the following brief but typical comparison:

Napierian System.	*Common System.*
Log 2·0 = 0·6931472	Log 2·0 = 0·3010300
Log 20·0 = 2·9957323	Log 20·0 = 1·3010300
Log 200·0 = 5·2983174	Log 200·0 = 2·3010300
Log 2000·0 = 7·6009025	Log 2000·0 = 3·3010300

and the reason for the unassailable superiority of the common system, with regard to calculation on the basis of counting in general use, will be obvious on consideration.

CHAPTER VI

RATIO, PROPORTION, VARIATION AND RATES

Ratio.—It is frequently necessary and convenient to compare two groups of things—*i.e.* to state, not their actual magnitudes, but the relation which the magnitude of one group bears to the magnitude of the other.

Thus, a group A comprising 20 oranges would be double of another group B comprising 10 oranges; and the statement might well be made that the group A "is to" the group B "as is" 2 to 1, the method of recording such a relation in practice being

$$A : B :: 2 : 1.$$

The convenience of this lies in the fact that if both groups be repeated to any number of times, the total number of oranges in the "A" groups will be to that in the "B" groups as is 2 : 1; and the ratio would hold also between the results obtained by taking the fifth, fortieth or "n-th" part of an A group and the fifth, fortieth or "n-th" part of a B group.

Now, although it may appear (and, indeed, is frequently stated) that ratio cannot be expressed between the contents of a group which comprises things of one kind and the contents of another group which comprises things of another kind, it should be noticed that we can, quite properly, compare two groups, one consisting of, say, dogs, and the other of, say, horses, if we are content to lose record of their identities, for the purposes of the com-

parison, by introducing the comprehensive title, say, "animal" to include both a dog and a horse. It is only necessary, in expressing a ratio, that the two groups shall consist of things which are *all* sufficiently similar *for the particular commercial purposes in view*; and, obviously, the degree of similarity must vary with the commercial purposes in different cases.

For instance, if we were under contract to supply a number of "animals," with no specification as to kind, a dog might count as one animal equally with a horse; and the degree of similarity would be low because the term is wide. On the other hand, a diamond merchant seeking a stone to match the "Koh-i-noor" in all respects would be working to an extremely high degree of similarity.

Complete similarity is, of course, impossible in this world; and it is of the greatest importance that students of engineering should appreciate both the fact and its effects, in commerce and in the processes of practical calculation. In counting groups of things—and, therefore, throughout the entire range of practical mathematics—we work on the basis of an appropriate "degree of similarity" among the things concerned; and in the affairs of real life, such degrees of similarity are always conventional, and never absolute.

The statement made above regarding the two groups of oranges was conditional upon the requisite degree of similarity being obtained. Provided that every orange in the group A were not only sufficiently like every other orange in that group, but also sufficiently like every orange in the group B, the oranges forming the group A could be arranged to make exactly two groups like B. This is the whole principle of ratio.

Having 28 similar things in two groups, one of 16 and the other of 12, the two groups might be "compared" according to their contents (*i.e.* the "ratio of their magnitudes" might be determined) thus: In the first place we might, of course, state the ratio as 16 : 12, without

reduction; and this would be the only course open if the two numbers had no common factors—as would be the case had the 28 things been arranged in two groups of 3 and 25, 9 and 19, 13 and 15, etc. As the two numbers have common factors, however, it is convenient to determine the "highest" common factor—*i.e.* 4—and to consider a subgroup of 4 things as a new "unit thing" for the purposes of expressing the ratio. Noting the fact that the first group comprises 4, while the second group comprises 3 of these new unit things, the ratio between the contents of the two groups may be expressed as 4 : 3. The converse is obvious.

Ratios of fractional quantities are important and useful. One instance will be sufficient to show that the underlying principle is the same for them as for groups comprising whole numbers of things.

Suppose it be required to express the ratio borne by $\frac{9}{16}$ of a thing to $\frac{7}{12}$ of a similar thing.

In the first group (called, for obvious reasons, the "antecedent") we have 9 things, each of which is one-sixteenth part of the unit thing; and in the other (called the "consequent"), 7 things, each one-twelfth part of the unit thing. The things of the one group are, then, sufficiently like each other; but they are not sufficiently like the things of the other group. Our first step must be to so manipulate them that the essential condition of ratio shall be fulfilled—*i.e.* that *all* the things concerned shall be sufficiently similar to permit of their being classified under a single denomination.

A little consideration of the case in point will show that 48 is the "least common multiple" of 16 and 12; and that if each sixteenth part be subdivided into 3 equal parts, each of these will be $\frac{1}{48}$ of the unit thing, while if each twelfth part be subdivided into 4 equal parts, each of these also will be $\frac{1}{48}$ of the unit thing. All the things concerned will then be sufficiently similar; and the ratio of the two magnitudes may be expressed.

Each of the things which formerly made up the first group having been divided into 3, there will be 3 times as many of the new things in this group as there were of the original things — *i.e.* the first group will now comprise $9 \times 3 = 27$ things. Similarly, the second group will comprise $7 \times 4 = 28$ things; and since 27 and 28 have no common factor greater than 1, the ratio cannot be expressed more simply than as 27 : 28.

If this line of argument be applied carefully to typical instances of ratio in arithmetic, algebra, trigonometry and other branches of mathematics the student will obtain common-sense views of the matter which cannot fail to be of great service in practical calculation.

Care should be taken to keep the essential condition —*i.e.* the adequate degree of similarity between *all* the things concerned — always clearly in view. Otherwise there is danger of being led into such common laxities as that which defines the modulus of elasticity as the "ratio" borne by a *stress* to the accompanying *strain*. Stress and strain are certainly not sufficiently similar to permit of a ratio being properly expressed between them;[1] and although one may be enabled to pass examinations by forcing oneself to dispense with a rational and common-sense interpretation of what is really a very simple matter, by thinking of numbers instead of things, such a course is highly undesirable. The ordinary man knows quite well that numbers have no significance except as records of counting real things; and if he accept results found from juggling with numbers, while feeling that he is ignorant of the things concerned, he will have no real

[1] A discussion of the modulus of elasticity on the basis of plain fact, showing that the "ratio of stress to strain" has no part or place in the significance of this useful conception, may be found in the author's book on *Structural Steelwork* (Longmans, Green & Co.), in which it is also shown that the moment of inertia and the section modulus are ordinary lengths, measurable in ordinary inches, and capable of physical visualisation easily by any one sufficiently interested.

confidence in the reality of those results, or in their applicability to the things of actual life. Lacking true confidence in his own power to command the things with which he is called upon to deal, he will inevitably either lose or fail to acquire proper self-respect—and the loss of self-respect is the one and only disaster which the real engineer dreads.

Proportion.—Proportion is the *equality of ratios.*

Thus, four magnitudes are said to be "proportional," or "in proportion," when the ratio borne by the first to the second is equal to that borne by the third to the fourth. For example:

5 dogs : 20 dogs : : 3 cats : 12 cats.

In the treatment of Proportion there arises a point to which the author would invite the most careful attention. This point is extremely simple; and the only thing in connection with it difficult to understand is that so obvious a point should have escaped notice. Yet the fact that it *has* escaped notice (or, at least, that if the point itself has been noticed, its full significance has not been grasped) is demonstrated unmistakably in many ways—notably by the deplorable vagueness which exists concerning that mysterious "conception" (and even the boldest have not ventured to define it more precisely than that) which is called "MASS."

As is well known, the "rule" commonly used in dealing with Proportion is: "The product of the extremes is equal to the product of the means"; or, expressed symbolically:

If $\quad a : b : : c : d;$

then $\quad ad = bc;$

and $\quad \frac{a}{b} = \frac{c}{d}.$

Now, let us examine the facts with a view to ascertaining the extent to which this rule may be justified.

Can we say that because a group consisting of 5 dogs "*is to*" another group comprising 20 dogs "*as is*" a

group consisting of 3 cats "*to*" a fourth group comprising 12 cats, *therefore* 5 dogs multiplied by 12 cats *is equal to* 20 dogs multiplied by 3 cats? Such a statement is obviously absurd, and totally unjustifiable.

If we are content to lose all record as to the individual characteristics of the things comprised in the four distinct groups, leaving the mere *numbers*, the rule gives

$$5 \times 12 = 20 \times 3 ;$$

or

$$60 = 60 ;$$

and it is for this reason alone that the "rule" was devised. The whole trouble has arisen from this unwarrantable divorcing of numbers from the things to which in fact they relate.

All that we are entitled to infer from the statement of a Proportion is the equality of the two ratios implied by it; and ratio, as we have seen, can only connect two groups of things when *all* the things concerned—*i.e.* the total contents of both groups considered as a single assemblage—besides being real, are sufficiently similar for the purposes of such comparison.

If there be given one group (A) of things, and we desire to arrange another group (B) to comprise things of the same kind, such that the new group shall bear to the given group the same relation (as regards the numbers of things comprised) as that which subsists between two other groups (D and C respectively), all that we are entitled—and all that we need—to argue is that:

> *The number of things in the group B must be equal to the number of things in the group A, varied in the ratio borne by the contents of the group D to the contents of the group C*;

which may be expressed symbolically thus:

$$N_B = N_A \left(\frac{N_D}{N_C}\right).$$

There is nothing vague or misleading about this

symbolism ; nor can it give rise to confusion or misunderstanding unless it be mutilated.

5 dogs = 20 dogs, varied in the ratio borne by 3 cats to 12 cats, is a perfectly plain statement of a real or realisable state regarding groups of things ; and there is no question involved in it, either of multiplication between unlike things, or change of nature in the things concerned.

The student will do well to consider a few typical examples—sufficient to satisfy himself completely as to the soundness of the argument advanced above—dealing with Proportion as applied in practical calculation. One such example we will discuss here, its supreme importance to all practical men, coupled with the vague and mistaken ideas so frequently expressed concerning it, warranting its consideration in some detail.

Force, and its measurement for practical purposes, is discussed in Chapter XII. ; but here we may note that we are only conscious of force through its motion-producing effects upon bodies to which it is applied, and that our measurement of it is based entirely upon the observed fact that, with regard to any particular body, the acceleration produced by the application of a force is directly proportional to the magnitude of the force applied.

Thus, if a_1 be the acceleration of a body produced by the application to it of a single unbalanced force of magnitude F_1, then the application to the same body of a single unbalanced force of magnitude F_2 would produce an acceleration a_2, such that

$$a_2 : a_1 :: F_2 : F_1 ;$$

or, conversely, if that body be observed to be moving with an acceleration a_2, it must be so moving under the influence of either a single unbalanced force of magnitude F_2, or under the influence of a system of forces having an unbalanced resultant of magnitude F_2, such that

$$F_2 : F_1 :: a_2 : a_1 .$$

Now, the basis of comparison on which all our measure-

ments of force and acceleration rest is supplied by the resultant attraction which is observed to draw all bodies (near the earth's surface) towards the centre of the earth, and the acceleration produced by that force.

The resultant force so applied to a body we call *its weight*—commonly denoted by W; and the acceleration produced (approximately 32·2 ft. per second per second) we call the *gravitational acceleration*—commonly denoted by g.

Hence, in estimating the magnitudes of forces applied artificially (*i.e.* otherwise than by direct gravitational action) to bodies, and the accelerations produced by those forces, we need only denote the particular force to be estimated as F; and the corresponding acceleration as a.

The proportion may then be stated as

$$\mathrm{F} : \mathrm{W} :: a : g; \quad \text{or} \quad a : g :: \mathrm{F} : \mathrm{W};$$

according as we desire to estimate a force from an observed acceleration, or an acceleration from a known force.

For the purpose of facilitating arithmetical computation, these proportions may be stated in the forms:

$$\mathrm{F} = \mathrm{W}\left(\frac{a}{g}\right); \quad \text{or} \quad a = g\left(\frac{\mathrm{F}}{\mathrm{W}}\right);$$

and for the sake of clearness as regards ideas, these statements may be rendered in words, thus:

> *If a body, of known weight* W, *be observed to move with an ascertained acceleration* a *ft. per second per second, the resultant unopposed force acting upon it, in the direction of the observed acceleration* a, *must be equal to the* WEIGHT *of the body, varied in the ratio borne by the observed acceleration to the gravitational acceleration;*

and:

> *If to a body, of known weight* W, *there be applied some other force, or system of forces, in such manner that the resultant unopposed force is of magnitude* F,

then the body will move, in the direction of the resultant unopposed force F, *with an acceleration equal to the* GRAVITATIONAL ACCELERATION, *varied in the ratio borne by the magnitude of the force* F *to the weight of the body.*

In the expression

$$F = W\left(\frac{a}{g}\right),$$

the symbol g cannot properly be separated from the symbol a; for these denote the two accelerations which, by their relation to one another, determine the ratio which the force under estimation must bear to that force which we know as the weight of the body.

Yet, by some strange means, the symbol g *has* been torn away from its sole legitimate companion, and applied to the WEIGHTS of bodies, regardless of the well-known and obvious fact that the gravitational acceleration of a body is entirely independent of, and unaffected by, the weight of that body. The outcome of this improper alliance of the realities represented by the symbols W and g has been termed the "*mass*" of the body whose weight is W —and in view of the totally unjustifiable procedure from which this mysterious "conception" arose, it is not surprising to find that nobody has ever succeeded in giving a precise and convincing account as to its nature and significance.

Clearly, had Proportion been taken and applied as the simple reality which it is in fact, such errors could not have arisen; and instances of this kind should convince the student that two of the most important things to be observed in the study and practice of mathematics are "Don'ts":

1. DON'T try to separate numbers from the real things to which they relate, and apart from which they have no mathematical significance; and,
2. DON'T try to imagine that a result obtained from false reasoning is necessarily mysterious or super-

natural. As a general rule, such results are merely silly—though silly things may sometimes have terribly serious consequences.

On the usual basis of reasoning, the proportion

$$a : b :: c : d$$

permits the inferences

$$(a+b) : b :: (c+d) : d\,;$$
$$(a-b) : b :: (c-d) : d\,;$$
$$(a+b) : (a-b) :: (c+d) : (c-d)\,;$$

and $$(ma+nb) : (pa+qb) :: (mc+nd) : (pc+qd)$$

where m, n, p and q are ordinary factors of repetition or reduction.

The practical utility of these inferences is, of course, in no way impaired by the foregoing interpretation of Proportion; while their deduction (which may be left as an exercise for the student) on the basis of physical reality is at once more simple and more instructive than on the ordinary basis of mere numbers from which all association with real things has been abstracted.

Variation.—In practical engineering, it frequently happens that two distinct groups of things vary together, a change in the total number of things comprised in one group either causing or accompanying a corresponding change in the other, so that the numerical relation between them remains constant. This is described as "one group *varying directly* with another."

Again, it frequently happens that an *increase* in one group causes or accompanies a *decrease* in the other, the numerical product remaining constant. This is described as "one group *varying inversely* with another."

A typical instance of "direct variation" is provided by the discharge of liquid from the outlet of a pipe or drain (through which the liquid flows under a constant pressure) with regard to the period of flow. Here we have two distinct groups of different things:

(1) A number of gallons or cubic feet of liquid ; and

(2) A number of seconds, minutes or hours of time ;
and the two groups vary together, a change in the former being proportional to the corresponding change in the latter provided the specified conditions be maintained.

Now, for some particular case, it might be convenient to state this symbolically as

$$D \propto t ;$$

where D is the " discharge " in, say, gallons, and t is the " time of flow " in, say, minutes ; and from measurement of the discharge during some observed period it might seem that the relation could be expressed in the form

$$D = kt ;$$

where k is some " coefficient," " constant " or " multiplier." Such is the common interpretation of proportional variation ; but it is quite obvious that no coefficient, constant or multiplier can change *minutes of time* into *gallons of liquid*—nor need we desire that it should.

A little consideration will show that " k " in this case is neither more nor less than the number of gallons of liquid discharged from the pipe or drain per minute ; and this, repeated to as many times as one minute is repeated in the period of discharge under consideration, gives the total number of gallons discharged in that period.

Thus, k is merely the *time-rate of discharge* for the particular pipe or drain in question under the specified conditions ; and if D is required in *gallons*, while t is in *minutes*, the time-rate of discharge must be expressed in *gallons per minute*. If D were required in, say, tons, and t were expressed in *hours*, k would have to be expressed in *tons per hour* ; but it would still be merely the time-rate of discharge.

Rates.—The introduction of the word " rate " in the foregoing discussion calls for comment ; and in passing to more detailed explanation, the obviously implied con-

nection between the words "Rate" and "Ratio" should be carefully noted.

In actually estimating the time-rate of discharge for such a case as that instanced above, we should not measure the liquid issuing from the outlet during a single minute only, for that would clearly be liable to inaccuracy from many causes. We should allow the discharge to continue for a sufficient number of minutes to provide a reliable indication of the general flow; and we should then apportion the total discharge equally to the minutes making up the period, thus taking for k the *average* discharge per minute as shown by observation and measurement covering a range sufficiently wide to allow for the ordinary and inevitable irregularities of flow.

This apportioning (or averaging) might be expressed symbolically thus:

$$k=\frac{D_t}{t};$$

where D_t denotes the total quantity of liquid discharged (in gallons) during t minutes; and it is necessary that the student should quite clearly understand that by $\left(\frac{D_t}{t}\right)$ we do *not* mean that we are dividing gallons by minutes, nor that we are expressing a *ratio* between two groups of dissimilar things. By $\left(\frac{D_t}{t}\right)$ we represent only the physical process which is actually implied—*i.e.* dividing the quantity of liquid discharged into *as many equal parts as there are minutes in the period concerned;* and regarding one such part as the *average rate of discharge per minute.*

Here, then, is the true mathematical significance of the word "Rate." It applies to two groups of things which vary with mutual dependence upon each other, and its numerical magnitude is determined in exactly the same way as though we were concerned with a proper "Ratio," except that we so arrange that the "consequent" shall be *one thing*; but inasmuch as we are not entitled,

by the facts of the case, to regard the resulting comparison as a ratio, we adopt the word "Rate."

It is imperative that we always specify the terms in which any particular "rate" is expressed. Thus, in the case considered above:

$$k = \text{(so many) } \textit{gallons} \text{ per } \textit{minute.}$$

If this practice be followed, no vagueness or confusion can occur; while, on the other hand, if the terms be *not* specified, the symbol employed becomes really meaningless.

The title "time-rate" indicates merely that the "consequent" of the rate is a unit of *time*. For the sake of practical convenience we often use "distance-rates" (*e.g.* so many telegraph-poles per mile along roads and railways); "surface-rates" (*e.g.* so many houses per acre in town-planning); "volume-rates" (*e.g.* so many gallons per cubic foot); "cost-rates" (so many oranges for a shilling); and a whole multitude of other rates, with many of which the student is doubtless intimately acquainted.

Much valuable information may be acquired from a careful consideration of the Latin Roots "Ratio" and "Ratum" and their general meanings, in connection with the foregoing discussion.

It is desirable that—at least in thought—some distinction be drawn between "simple" variation—*i.e.* variation in which the quantities concerned occur in their first powers only, as in the case considered above—and variation in which one of the quantities varies directly (*i.e.* increase for increase) with the *square* (or some other power expressed by a positive index) of the other. There is no need to invent special names for such classes of variation. Precise diction will follow naturally—and to the properly appropriate degree of precision—if the thoughts and ideas to be expressed are sufficiently clear. The person who says that he knows exactly what he means, but cannot "put it into words," may be perfectly honest as regards intention;

but he is at least mistaken, for every thought or idea concerning the real things and experiences of actual life, *if based upon a clear and true perception,* can be expressed and described in simple words by any one who will take the necessary trouble—and it would be difficult to over-estimate the value, to an engineer, of the ability to express his thoughts and ideas completely in simple words.

Examples of Inverse Variation need not be discussed here. They follow exactly the same argument, in principle, as that outlined above in connection with Direct Variation.

It is, however, both interesting and instructive to examine the facts upon which the statements of inverse variation in practical work are based.

This may be left to the student as a highly profitable exercise; and it may be of assistance to suggest that, as a rule, he will find the fundamentals are those of *direct* variation, the inverse variation being introduced to serve practical convenience.

For example, it is commonly stated that the intensity of illumination thrown upon a surface by a given source of light varies inversely as the square of the distance between the surface and the source of light. This is, of course, a highly convenient way of expressing the results of observation; but the fundamental fact is that, by moving a particular object to a greater distance from a given source of light, the number of light rays falling upon it are reduced. Some rays which formerly fell upon it when nearer to the source now pass by it; and this *loss* of illumination varies with the distance between the object and the source of light.

It may appear at first sight that it matters little whether we work upon the basis of the *loss* or of what remains to us after the loss has been sustained, so long as the results of our calculations be correct and practical; and to some extent this is true—indeed, inasmuch as the latter course is usually the more direct of the two alternatives, it is preferable that it be used rather than the former, unless

the calculation in hand be concerned with *losses* of illumination. There is, however, the danger that such methods may give rise to erroneous inferences if the facts be not clearly understood.

For instance, suppose that a certain lamp be found to give an intensity of illumination equal to C standard-candle-power at a distance of 1 foot from an object. We might then say that at a distance of 2 feet the intensity of illumination would be $\frac{C}{2^2}=\frac{C}{4}$ candle-power; at 3 feet the intensity would be $\frac{C}{3^2}=\frac{C}{9}$ candle-power; and so on. But we could not properly say that at *no distance* the intensity would be $\frac{C}{0^2}=\frac{C}{0}=$ an infinity of candle-power! Several awkward inferences would follow such a statement; some of which might be very difficult to reconcile with fact.

All that we really mean is that at *any distance* between the source of light and the object there will be a loss of intensity in the illumination; whereas, at *no distance* there would be *no loss*. We know nothing of the actual illuminating power of a lamp, candle or other source of light at no distance from it; but it is clearly not infinite for *all* lamps, candles or other sources of light, irrespective of the quantities of energy transformed into light by them respectively. For practical purposes, we compare them all on the basis of a standard candle at a distance of 1 foot from the object which it illuminates; and we are not entitled to press the "law" indicated by the symbolism beyond the range over which we can observe its operation.

The sole object in calling attention to this point is that the student may be led to think clearly for himself regarding the fundamental facts of his work—to see through and beyond the mere symbols of an *expression* to the realities upon which it rests, rather than to follow the all-too-common course of regarding the symbols as supernatural despots which *dictate* laws instead of obeying them.

Another point of importance in this connection is the proper allocation of "loss-coefficients" and similar factors to the terms with which they are physically related. Many a doubt, difficulty and discouragement has been caused by carelessness in this respect.

Consider the flow of liquids through pipe systems, as a typical instance.

If the liquid head be h feet, the theoretical velocity of flow will be

$$v = \sqrt{2gh}\,;$$

where g is the gravitational acceleration in feet per second per second, and v the velocity of flow in feet per second.

If the internal diameter (or "bore") of the pipe be d feet, the cross-sectional area of the flow, running full bore, will be

$$A = 0{\cdot}7854 d^2\,;$$

and every second, were it not for losses due to frictional and other resistances, a rod (so to speak) of liquid, A square feet in cross-sectional area, and v feet in length, would emerge from the outlet of the pipe as discharge. The time-rate of discharge, in cubic feet per second, might therefore be expressed as

$$\Delta_{(\text{SEC.})} = 0{\cdot}7854\, d^2\sqrt{2gh}.$$

This, however, is only the theoretical estimate, disregarding all such disturbing influences as the retardations caused by roughness of the internal surfaces of the pipe; by changes in the direction of the flow; by molecular friction in the liquid itself; by bubbles of air or vapour; and so on.

The extent of these retardations will depend upon the bore and length of the pipe system; the kind and quality of the material from which it is made; the number and sharpness of the bends and elbows; the care and skill exercised in manufacture and installation; the viscosity and physical condition of the liquid—in short, upon so

many and various factors that it would be useless to attempt an estimate based upon logical reasoning alone. The practical alternative is to measure the actual discharge for an observed period; and to express this as a fraction of the theoretical discharge. This fraction is commonly termed a "loss-coefficient"; and, denoted by ϕ or some such symbol, is introduced into the expression for $\Delta_{(\text{SEC.})}$, thus:

$$D_{(\text{SEC.})} = \phi . \Delta_{(\text{SEC.})} = \phi\{0 \cdot 7854 d^2 \sqrt{2gh}\}.$$

Now, if the pipe does in fact run full bore—or as nearly so as can be seen for practical purposes—it is obvious that the loss of discharge cannot bear any relation to the cross-sectional area of the pipe. Evidently, the loss is due to retardation of the flow; and hence, the actual effective velocity of flow is not $\sqrt{2gh}$, but $(\phi\sqrt{2gh})$. Moreover, since h represents a definitely measurable height, the retardation must be due to the actual acceleration acquired by the descending water in the pipe being less than that which would be acquired by a body falling freely, in a vacuum, under the influence of the gravitational acceleration—which, after all, is a fairly obvious conclusion. The only proper course, therefore, is to combine ϕ with g in such manner as to indicate a velocity of flow in accordance with the ascertained facts; and seeing that the velocity varies with the square root of the acceleration, ϕ must be squared for application to g in the expression, which then becomes

$$D_{(\text{SEC.})} = \{0 \cdot 7854 d^2 \sqrt{2(\phi^2 g)h}\}.$$

It is not strictly consistent to call ϕ a "loss-coefficient," for it clearly relates, not directly to the loss, but to that which remains after the loss has been sustained, leaving the real "loss-coefficient" as $(1-\phi)$. Combining this with g as ϕ was combined above for $D_{(\text{SEC.})}$, the discharge (in cubic feet per second) *lost* through retardation will be

$$\delta_{(\text{SEC.})} = 0 \cdot 7854 d^2 \sqrt{2\{(1-\phi)^2 g\}h} =$$
$$0 \cdot 7854 d^2 \sqrt{2\{(1-2\phi+\phi^2)g\}h}.$$

Obviously, if the actual discharge be added to the lost discharge, the sum will be equal to the full theoretical discharge, *i.e.*

$$D_{(\text{SEC.})} + \delta_{(\text{SEC.})} = \Delta_{(\text{SEC.})};$$

and if the corresponding expressions be examined, it will be found that they fulfil this condition.

The student will find that clear-thinking with regard to such matters will be the means of preventing him from being misled by many common fallacies—notably the confusion of a "factor of safety" with the provision of an adequate *margin for contingencies.*

Nature of Rates.—A "rate" represents a group of real things, just as do all other mathematical symbols or expressions.

If we find that liquid flows from the outlet of a pipe at the *rate* of 3 gallons per minute, the discharge is simply 3 *gallons of liquid.* It is not altered in any way by reason of the fact that it emerges from the pipe in a minute of time; and if we sold the liquid at so many shillings per minute of flow, on the agreed basis that the rate of flow was 3 gallons per minute, the purchaser of one minute's flow would expect, and be justly entitled, to receive 3 gallons of the liquid. If he paid for 17 minutes' flow—*i.e.* 17 repetitions of one minute's flow—he would be entitled to receive 3 gallons of the liquid repeated to 17 times.

This point is of the utmost importance in many and various fields of practical calculation. Failure to appreciate it has led to many errors—notably, that the Moment of Inertia is "four-dimensional,"[1] and Section Modulus "three-dimensional."

In dealing with the Differential and Integral Calculus (see Chapter XI.) it will be found that we are almost entirely

[1] For a demonstration of the fact that Moment of Inertia and Section Modulus are both simple lengths—each the arm of an equivalent couple, see *Structural Steelwork,* by Ernest G. Beck (Longmans, Green & Co.).

concerned with "rates." In Differentiation we determine *rates of variation* from known data concerning the constant relation borne by the total contents of one group of things to the total contents of another group of things; and in Integration we determine the change in the total contents of one group brought about by the other group varying through some specified range, it having been established that the *rate of variation,* while not necessarily constant, follows a definite law or order.

Clearly, then, to obtain a really workmanlike control of this important branch of the subject, it is necessary that "rates" should be thoroughly understood, without the slightest vagueness as to their nature, purpose and influences. The student is strongly advised, therefore, to satisfy himself that he has acquired a real and comprehensive perception of the principle which underlies "rates" and their manipulation. The principle is, as we have endeavoured to show, perfectly simple and practical; and there is not the slightest desire to suggest that there is any difficulty to fear in connection with it. In most cases, the author has found that the greatest difficulty encountered by students is in regarding the matter from the standpoint of actual physical fact, without any obstructions in the way of "abstractions" or "mathematical conceptions." Granted the right attitude and spirit, however, this difficulty is soon overcome; and the rest follows easily.

Except as a matter of interest, and as an example on which to test one's powers when a sound practical knowledge has been acquired, no effort need be wasted upon such statements as that: "Force is the *space-rate of change of work*; and also it is the *time-rate of change of momentum.*" What is needed is a clear understanding of such rates as an engineer uses in his daily practice, in dealing with the affairs of ordinary life.

CHAPTER VII

SIMPLE AND SIMULTANEOUS EQUATIONS

Equations of the First Degree.—Much of the work relating to equations becomes obvious if the expressions be regarded less as mere statements of static equality than as relations borne by what may be termed the "arrangement-specification" of an assemblage to its total numerical contents.

For instance, given the equation

$$7x = 168,$$

the particularisation of x becomes an extremely simple matter if we notice that on the left-hand side of the equality sign we are told that the things forming the group under consideration are (or may be) arranged in 7 *rows, with x things in each row*; while on the right-hand side we are told that the total number of things in the group is 168. It is, then, but natural that we should arrange the 168 things in 7 equal rows—the operation being readily performed on the lines adopted in dealing cards for Whist or Bridge, except that the number of "hands" may not always be 4, and also except that a few things may have to be divided into parts in order that the "arrangement-specification" may be fully complied with where the total number of things is not an integral multiple of the specified number of "hands" or rows. The number (evidently, 24) of things in each of these rows is, clearly, the number indicated by x; and its determination "solves" the equation.

With an equation of the type

$$4x+9=53$$

the "arrangement-specification" indicates unmistakably that the group consists of two subgroups—one comprising $4x$ things, arranged in 4 rows with x things in each row (this subgroup being, therefore, dependent upon x); and the other comprising 9 loose things (this subgroup being independent of x).

A little consideration will show that each of the 9 loose things must be sufficiently similar to each of the $4x$ things to permit of the whole assemblage being counted under a single denomination for statement as "53 things"; and it is important that the student should realise and appreciate the fact that the same condition applies to *every equation*, irrespective of its "degree," and whether it be applied either singly or in simultaneous combination with others.

To solve this equation by physical means, then, we might place 53 similar things on (say) a Halma board; and, removing 9 things from the group, arrange the remaining 44 in 4 equal rows. The number (11, clearly) of things in each of these rows is, of course, the "solution" of the equation.

Many convenient devices on similar lines will doubtless present themselves on consideration; and there is no need for further detailed explanation as to the physical reality of the processes employed.

It is, however, both interesting and instructive to notice that all equations of the type

$$4x+9=53$$

may be conveniently generalised, leaving the relationship expressed by a single statement in terms which may be particularised according to the facts and conditions of individual cases as they arise.

First, it is clear that the "arrangement-specification"

$4x+9$ may completely cover a vast range of groups, according to the number of things (denoted by x) to form each of the four rows, even with 9 loose things in each individual group. If x were 1, for instance, the group would comprise 13 things in all; if x were 47, the total assemblage would number 197; if x were 0, the group would consist of the 9 loose things only; and so on.

Moreover, in some cases it is necessary to deal with *bills demanding payment of things*, and also with *vouchers entitling the holder to receive things*, instead of (or besides) actual things, in exactly the same way as we dealt with such bills and vouchers in Addition, Subtraction, etc. (Chapter II.); and hence, x may be regarded as positive in some cases, and negative in others. For instance, it is often convenient to denote back pressure upon the piston of an engine as *negative pressure*; and to denote upward loads (not mere reactions, but definitely active loads) applied to a beam as *negative loads*, in contradistinction from the ordinary downward gravitational loads which are regarded as positive.

For this reason alone, then, the statement $4x+9=53$ should be regarded as only a particular case of the general type, which might be expressed as

$$4x+9=y.$$

Further, it may be convenient to deal with assemblages of similar formation, but comprising *any* number of equal rows of x, and also *any* number of loose things, instead of 4 and 9 (respectively) in particular.

Hence, a general expression, in the form

$$ax+b=c$$

may be used to denote the relation between the "arrangement-specification" $(ax+b)$ and the total number of things (c) comprised, for *all assemblages of things* capable of arrangement in some number (a) of *rows* with x things

in each row, leaving some number (b) of things loose—*i.e.* apart from the uniform rows of x.

On particularisation of a and b, the expression shows the relation between the arrangement-specification and the total numerical contents for *all* assemblages capable of arrangement in a rows with x things in each row, leaving b loose things over; and on particularisation of x, the total number of things comprised in a certain individual group are determined.

It is important to notice that, in such an expression as

$$ax + b = c,$$

the symbol a (sometimes called the "coefficient of x," but more properly the *number of rows*—or the number of times to which one row of x things is to be repeated) indicates the *rate* at which the total contents of the assemblage vary with respect to x.

This will be clear from observation of the fact that the addition of *1 thing in each row* has the effect of increasing the total contents of the group by a things—since there are a rows, each of which is increased by 1 thing; from which it follows that the total contents of the assemblage vary numerically at the rate of *a things per unit variation in the number of things in each row*—that is, in x.

Further, the symbol b in the expression (sometimes called the "constant" term, in contradistinction from ax, which is then called the "variable" term) indicates a number of things (each of the same kind as those forming the subgroup indicated by ax) which are not affected, numerically or otherwise, by any change in x. Thus, b is *constant* only in that it *does not vary with x.* It is not one of those mysterious and elusive things which are so often called "constants" more for the sake of peace and quietness than for any more logical reason, and which have so little of real constancy in their composition that we find them at one moment "pure numbers," at another "pounds of force," at another "feet per second per second," and

so on, presumably in accordance with the humidity of the atmosphere, or the trend of the political situation at the moment.

Of course, both x and b may denote either actual asset things, vouchers authorising the holder to account himself the owner of things not actually to hand, or bills calling upon the holder to deliver actual things either immediately or later ; and a may denote repetitions, divisions or withdrawals of actual things, vouchers or bills. But all through the expression, only one kind of *thing* is concerned.

For instance, if we were dealing with groups of sheep in connection with some set of transactions, x might be $+14$, and b might be -3 ; in which case the contents of the assemblage could be expressed as

$$a(+14)+(-3)=c.$$

If a were 5, the contents of the group would be

$$c=5(+14)-3=70-3=67 \text{ sheep} ;$$

and the facts recorded would be that the owner of the group stood possessed of 70 sheep (in 5 groups of 14), but had contracted a liability to deliver 3 sheep, thus reducing his actual assets to 67 sheep.

Or, x might be (-8), and b $(+27)$; in which case, if a were still 5, the contents of the group would be

$$\begin{aligned} c&=5(-8)+27=-40+27 \\ &=-13 \text{ sheep}, \end{aligned}$$

indicating that the owner of the group in question had assets amounting to 27 sheep, but had incurred liabilities amounting to 40 sheep (in 5 separate bills, each demanding the delivery of 8 sheep) ; and on paying away all his asset sheep, he still remains in debt to the extent of 13 sheep.

Or, again, x might be (-27), and b (-2), while a is $(-\frac{2}{3})$; in which case, the contents of the group would be

$$\begin{aligned} c&=\{(-\tfrac{2}{3})(-27)-2\} \\ &=+18-2=16 \text{ sheep}. \end{aligned}$$

The facts recorded in this statement might be that the owner of the group held bills calling upon him to deliver 2 sheep, and other bills requiring him to deliver 27 sheep; but these latter bills were subject to a discount of one-third of their face-value, leaving the number of sheep demanded by them $\frac{2}{3} \times 27 = 18$.

Now, the negative sign before the $\frac{2}{3}$ denotes that this parcel of bills is *withdrawn* (or totally discounted); and hence, the 18 sheep which the owner of the group under discussion had formerly been required to consider as belonging to some one else, now belong properly to him—thus becoming either: (1) an increase of his assets (if he had assets, apart from the group in question, wherewith to pay the bills); or (2) a corresponding reduction of his liabilities (if he had no such assets); and in either case, to be counted as positive with regard to the group in question.

It may be acceptable, as an amplification of the treatment of the "Rule of Signs" in Chapter II., if we consider this case a little more fully.

Let us suppose that the man possesses, apart from the group represented by the statement $\{(-\frac{2}{3})(-27)-2\}=3$, asset sheep numbering A; and liabilities to the extent of L sheep. Then, the state of his affairs might be recorded as

$$\mathrm{A}-\mathrm{L}+\{(-\tfrac{2}{3})(-27)-2\}=\mathrm{A}-\mathrm{L}+c;$$

or

$$\mathrm{A}-\mathrm{L}+\{-(-18)\}-2=\mathrm{A}-\mathrm{L}+c.$$

Now, the symbols within the music brackets indicate that the man is relieved from the necessity for paying away 18 sheep; and hence, his liabilities are reduced from L to (L − 18).

Thus the statement becomes

$$\mathrm{A}-(\mathrm{L}-18)-2=\mathrm{A}-\mathrm{L}+c;$$

but the effect would be the same if, instead of withdrawing the bills, he were given either 18 real sheep, or a voucher entitling him to account himself the owner of 18 sheep

not actually to hand at the moment—thus increasing his assets instead of reducing his liabilities.

Hence, the effective state of his affairs may be expressed as

$$A+18-L-2=A-L+c\,;$$

whence, obviously,

$$\begin{aligned}\{(-\tfrac{2}{3})(-27)-2\} &= +18-2=c\\ &= +16 \text{ sheep.}\end{aligned}$$

It is sometimes found rather difficult to understand why the withdrawal of a bill from a group should have the effect of increasing the assets of that group—in other words, why a bill cannot be just torn up or otherwise destroyed, and that be the end of the matter, so that the liability represented by the bill would simply disappear, instead of being turned into an asset. The explanation lies in the fact that we are concerned always with *real things*, either as assets or liabilities ; and further, although both sides of every equation relate to *one and the same group* (the difference between the two sides being nearly always that whereas one indicates a particular arrangement-specification, the other indicates another arrangement-specification of the same assemblage), there is, with positive and negative signs involved, always a *second* group (or store) *implied*—a group which, as a rule, we do not need to number as we do that specified by the particular equation under consideration ; but which has the power to give out an increase of assets to the equation-group, in place of withdrawing bills due to it (the implied group, or store) from the equation-group.

The number of things concerned in an equation, then, whether they be actually at hand or the equivalent in vouchers or bills, remains unchanged throughout the process of solution. If we destroyed either a bill or a voucher, its significance would vanish—which means that the *real things* which it represents would *cease to exist*, so far as both the equation-group and the implied group (or store)

are concerned. Even if we imagined them simply removed, they would pass to a third group, of which we have no knowledge or record—and we are not permitted to so interfere with the total number of real things concerned in any mathematical equation or expression whatsoever; for real things do not vanish in fact, nor can they be properly lost in transit between the two groups with which they are in fact concerned. If by some means it be made to appear that they have so vanished or been lost, they will be found, always, to have been merely misplaced—which all too frequently means that they have been acquired by some one who has no right to them.

These matters are well worthy of careful consideration; and if the student will take the trouble—only trouble, and not very much of that, is required—to acquire a full and clear knowledge of the underlying realities, he will soon find that he has learned many things, besides what is commonly called Mathematics. If the man-in-the-street would study accountancy on the lines indicated in the preceding paragraphs, and apply the results of his deliberations to the ordinary transactions of his daily life, we should seldom see the anomaly of one man growing fat in idleness while a thousand others, who have worked hard all their lives, die of starvation and dirt.

Two examples of engineering calculation, both of which might be represented by the equation

$$(-\tfrac{2}{3})(-27)-2=+18-2=+16,$$

may be of interest.

First, suppose we were estimating the *upward* forces, in pounds, acting upon the piston indicated in Fig. 45.

The conditions represent a downward (*i.e.* a *negative upward*) force of 2 lb. due to the weight of the piston and rod; while there is evidently a suction (or *negative pressure*) on the upper (or negative) surface of the piston.

The truth of the argument, and of the solution obtained above, will be apparent. If the piston is to be held

stationary a weight of 16 lb. must be applied, to prevent it from rising.

Second, suppose we were estimating the rotational tendency, in foot-pounds, acting upon the hinged lever indicated in Fig. 46.

The weight of the lever is 2 lbs., and its centre of gravity at C, 1 ft. to the left of the hinge O.

Suppose that we agree to regard clockwise rotation about O as *positive*. We must, then, regard anti-clockwise rotation about O as *negative*; and to meet this agreement, we must adopt the convention that downward forces shall be regarded as positive forces, while upward forces are negative; and leverages to the right of O are positive leverages, while leverages to the left of O are negative.

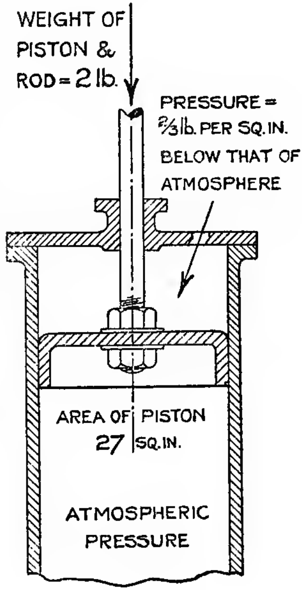

FIG. 45.

To prevent actual movement of the lever as a whole

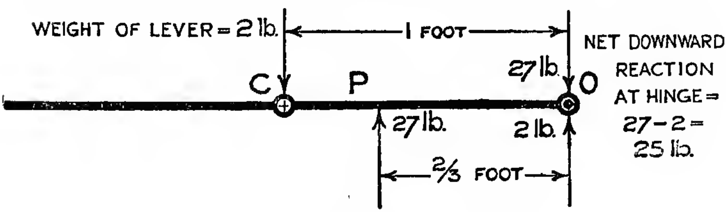

FIG. 46.

(mere rotation will, of course, allow one point in it to remain stationary), there must be provided at O an upward reaction of 2 lbs. to balance the weight of the lever; and

also a downward reaction of 27 lbs. to balance the load of 27 lbs. applied at P. Thus, the lever will be subjected to the action of two couples: one producing an anti-clockwise (*i.e.* negative) rotational tendency of 2 lbs. at 1 ft. = 2 ft. lbs.; and the other producing a clockwise (*i.e.* positive) rotational tendency of 27 lbs. at $\frac{2}{3}$ ft. = 18 ft. lbs.

According to the convention adopted, both leverages are negative; the load of 2 lbs. is positive; and the load of 27 lbs. is negative.

For this case also, the solution obtained above is clearly true. To hold the lever in equilibrium, a rotational tendency of 16 ft. lbs., anti-clockwise in sense, must be applied.

The Linear Relation.—The expression

$$ax + b = c$$

is frequently spoken of as typifying the "linear relation" or "straight line law."

This is a convenient name for the expression, and there is no strong reason to be urged against its use; but it would seem that many who so speak of it do so entirely because it gives a straight line if plotted upon squared paper. If an equation of this type were plotted as a "graph" with relation to two axes, either or both of which were curved, the line representing the variation would, of course, be curved; and, even apart from this, few would find real satisfaction in the statement that the relation is a "straight line law" with no firmer basis than the fact that one means of expressing (or representing) it gives a straight line (see Chapter IX.).

The author would suggest that there is a better reason for the title in the fact that the arrangement-specification provides that the original unit thing concerned shall first be repeated to x times to form a *row* or *line*; and since a row may be either curved or straight according to convenience, while the straightness or curvature of the row does not in any way violate the arrangement-specification, it is better that the relation be thought of as the "linear

relation," without unnecessary and irrelevant restriction as to straightness.

In the language and symbolism of the Differential Calculus, a (which, as we have seen, denotes the *rate* at which the total contents of the assemblage vary numerically with respect to x) would be called the "differential coefficient of c with respect to x"; and would be written

$$\frac{dc}{dx}=a.$$

Conversely, knowing that the total contents of a group vary at the rate of a things per unit variation in x, we know that the assemblage must always comprise a number of things capable of arrangement in a rows, with x things in each row; whence, the relation between x and that portion of the assemblage which varies with x at the stated rate must be capable of representation in the form

$$ax=k.$$

Since, however, if there were some number of loose things in the group which did not vary with x, such things would not affect the rate of variation $\frac{dc}{dx}=a$, we must provide for the inclusion of such loose things by writing the "arrangement-specification" as

$$ax+b=k+b=c.$$

This is the principle known as Integration; and a perusal of Chapter XI. will show that there is very little in the "calculus"—as regards principle, at least—with which the ordinary man is not thoroughly familiar, and nothing in its application to the calculations and investigations of practical engineering which need cause the average student any fear or misgivings.

Obviously, there is no need to apply the methods of the calculus to groups which follow the linear relation, for the rate of variation in such cases may be determined by mere inspection; while, conversely, knowing the rate

of variation to be simply a constant number (*i.e.* independent of x) of things per unit variation of x, the "arrangement-specification" for the group may be written, in general terms, straight away.

With other relations, however, the methods of the Calculus are necessary; for in some cases the rate of variation and the "arrangement-specification" for the group would be impossible—and in many extremely difficult or tedious—to determine by other means.

Nevertheless, the principles involved remain always based upon actual and simple physical processes, demonstrably real or realisable; and the symbolism throughout relates to groups of real things, while the argument is as plain and straightforward as in the case of the linear relation described above.

Simultaneous Linear Relations.—It sometimes happens, in practical calculations and investigations, that a group of things is required to comply with two distinct arrangement-specifications at the same time—in which case the two equations are said to be "simultaneous"; though what is really meant is that the required solution must satisfy both equations at once.

Let us take first the simple case:

$$\begin{cases} 2x + 5 = y \\ 3x + 1 = y. \end{cases}$$

We have one group of things capable of arrangement in 2 rows of x, with 5 loose things over; and another group of things capable of arrangement in 3 rows of x, with 1 loose thing over. Clearly, the total number of things comprised in the one group is equal to that in the other group—since both are capable of arrangement in a single row of y things. Clearly, also, the things comprised in one group must be of the same kind as those in the other group, since the particular values of x and y known as the "solution" must relate equally to both equations.

If we imagine two groups, each comprising the same

total number of things, but in one the assemblage arranged in 2 equal rows (the number of things in each row being x, which has yet to be determined) with 5 loose things over, and the other in 3 equal rows—equal, that is, not only to each other, but also to each of the two rows of the former group—with 1 loose thing over, it is but natural to remove one loose thing from each group, leaving the first with 4 loose things, and the second with none. Further, if we remove 2 complete rows of x from each group, the first will comprise only its 4 loose things; while the second will consist solely of its one complete row of x things. Since the groups were numerically equal at the outset, and only equal withdrawals from both have been effected, the remainders must be equal—whence, to satisfy both arrangement-specifications, the one complete row of x must consist of 4 things.

Then, whether we collect an assemblage according to the first arrangement-specification or the second, the total number of things (y) comprised will be 13; giving the solution

$$x=4\,;\quad y=13.$$

Now let us take, as a second case, the pair of simultaneous equations

$$\begin{cases} x-3=2y \\ -2x+21=6y. \end{cases}$$

Here we have specified two groups of similar things, one capable of arrangement in 2 rows of y, and the other in 6 rows of y. Also, the first group is capable of arrangement in 1 row of x, provided we include a bill for 3 things, retaining the net assets as 3 things less than one complete row of x; while the second comprises 21 loose things, with a bill demanding payment of $2x$ things.

Suppose, now, that we double the first group, so that it will contain 2 rows of x, with bills for 6 things; its total contents being capable of arrangement in 4 rows of y—so that the relation between arrangement-specification and total contents is preserved.

If we add both assemblages together to form a single group, pooling the assets and liabilities of both, the two complete rows of x things may be paid away in discharge of the bill for $2x$ things; while 6 of the 21 loose things may be paid away in discharge of the bill for 6 things, leaving only 15 loose things forming the total assemblage. But it is provided that the total contents of the single group shall be capable of arrangement in 10 (*i.e.* $6+4$) rows of y; and if this arrangement is to be effected, it is clear that each of the 10 rows will comprise $1\frac{1}{2}$ things—giving $y=1{\cdot}5$.

Clearly, then, 2 rows of y will comprise 3 things; and, from the first equation, x denotes a number of things such that, after paying away 3, the remaining things number 3—from which it follows that x is 6, giving the solution

$$x=6\,;\quad y=1\tfrac{1}{2}.$$

A little consideration will show that the ordinary methods in common use for solving simultaneous equations of the first degree with two "unknowns"—*i.e.* the methods generally referred to as "Elimination" and "Substitution" and a combination of these two—are completely covered by the arguments outlined in dealing with the two examples above.

It is very seldom that cases arise—and still more seldom that such cases need arise—in practical work, where simultaneous equations involving three or more "unknowns" call for solution. As an exercise, however, and for the purpose of clearly visualising the physical processes concerned, the student should analyse a few of such examples, to be found in any good text-book.

There is another method for treating simultaneous equations which the author has found helpful to some students; and which is convenient in some of the cases which arise in practical calculation. It is based upon the rates at which the total contents of the two groups vary with respect to x; and the rate at which, in consequence,

they converge towards the state of simultaneous satisfaction of the two arrangement-specifications.

For instance, with the pair of equations

$$\begin{cases} 2x+5=y\ ; \\ 3x+1=y\ ; \end{cases}$$

both are expressed in the typical form of the linear relation; and since the total contents of the first vary at the rate of 2 things per unit variation in x, while the second varies at the rate of 3 things per unit variation in x, it is evident that, from a net difference of 4 things when x is eliminated, the groups *converge* at the rate of $3-2=1$ thing per unit *increase* in x. Hence, to make up this difference (and so determine the state of simultaneous compliance with the specified relations) x must *increase* from 0 to 4; giving $x=4$, from which the value of y (*i.e.* 13) follows obviously.

Again, with the pair of equations

$$\begin{aligned} x-3&=2y\ ; \\ -2x+21&=6y\ ; \end{aligned}$$

expressing both in the typical form of the linear relation

$$\begin{cases} \tfrac{1}{2}x-1\tfrac{1}{2}=y \\ -\tfrac{1}{3}x+3\tfrac{1}{2}=y \end{cases}$$

it is clear that the net difference between the two groups when x is eliminated is $3\tfrac{1}{2}+1\tfrac{1}{2}=5$ things. Also, it is clear that the first group *increases* at the rate of $\tfrac{1}{2}$ a thing per unit increase in x; while the second *decreases* at the rate of $\tfrac{1}{3}$ a thing per unit increase in x. Hence, from the state in which x is eliminated, the two groups *converge*, from a difference of 5 things, at the rate of $\tfrac{1}{2}+\tfrac{1}{3}=\tfrac{5}{6}$ a thing per unit increase in x; and hence, the difference will be reconciled (*i.e.* the simultaneous compliance with the specified relations will be effected) by an increase in x from 0 to $(5\div\tfrac{5}{6})$—*i.e.* from 0 to 6—thus indicating the solution $x=6$, from which the corresponding $y=1\tfrac{1}{2}$ follows obviously.

An instance of the form in which these equations may arise in practical calculations is supplied by the example :

$$\begin{cases} 1{\cdot}34x+0{\cdot}87=y \\ -0{\cdot}71x+4{\cdot}09=y. \end{cases}$$

The total contents of the first group *increase* at the rate of 1·34 thing per unit increase in x, while those of the second *decrease* at the rate of 0·71 thing per unit increase in x; whence, from a net difference of $4{\cdot}09-0{\cdot}87=3{\cdot}22$ things when x is eliminated, the two groups *converge* at the rate of $1{\cdot}34+0{\cdot}71=2{\cdot}05$ things per unit increase in x. To make up the difference, then, x must increase from 0 to $(3{\cdot}22\div2{\cdot}05)$—*i.e.* from 0 to 1·57 ; giving $x=1{\cdot}57$, from which it follows that

$$\begin{aligned} y&=4{\cdot}09-(0{\cdot}71\times1{\cdot}57)=4{\cdot}09-1{\cdot}115\\ &=2{\cdot}975\,; \end{aligned}$$

the complete solution being

$$x=1{\cdot}57\ ;\ \ y=2{\cdot}975.$$

This method of argument will be found, on consideration, to cover some of the "rules" in common use ; and the student will benefit from reflection upon the fact that all the "rules" and "methods" acceptable in practical work are based upon—and derived from—solutions of the various types of problems by physical processes. As to the reason why the investigators should have so uniformly recorded their findings in the form of crystallised "rules" and "methods," instead of describing the simple processes on which they were deduced, we need not inquire—doubtless there was a good reason, or the course would not have been so generally adopted for centuries on end.

CHAPTER VIII

QUADRATIC EQUATIONS AND "IMAGINARY" QUANTITIES

Equations of the Second Degree.—A group or assemblage formed by first repeating an original unit thing to x times to form a row, and then by repeating each thing in that row to x times to form a square-group layer, may, following the lines of argument adopted in the preceding pages, be regarded as a group of the *second order*, or *second degree*; and an equation relating to such a group or assemblage is commonly known as a "Quadratic" equation—the suitability of the term being fairly obvious.

Like equations of the first degree, quadratic equations are merely statements as to the relation existing between the "arrangement-specification" of an assemblage of real things and its total numerical contents; and they should be regarded less as statements of static equality, and more as particular instances of the general expressions denoting such relations.

The arguments outlined in Chapter VII., regarding the standard and typical forms which cover the entire range of the linear relation, will need but slight modification to render them equally applicable to quadratic equations; and the development of these arguments may be left to the student as an exercise which will yield a handsome return for the time and trouble expended upon it. There should be no difficulty in the matter if it be approached and followed through completely on the basis of real things in groups, both "sides" of an equation relating always to the same group.

It will be found that the general expression of the quadratic equation may occur in two forms :

$$(1)\ ax^2=c\ ;\ \text{and}$$
$$(2)\ ax^2+bx=c\ ;$$

according as the group is either (1) entirely of the second order ; or (2) partly of the second, and partly of the first order.

Also, it will be seen that, as we thought of the "simple" equation in its general forms as typifying the "linear" relation, so we may think of the quadratic equation in its general forms as typifying the "superficial" relation.

Pure Quadratics.—Let us consider first that form of the quadratic arrangement which is entirely of the second order—sometimes called the "Pure" quadratic.

Taking first the simple case

$$3x^2+12=120,$$

the statement as a whole indicates clearly that, of an assemblage of 120 things, 108 are to be arranged in 3 layers, each layer comprising x rows, with x things in each row, leaving 12 loose things over ; and the equation will be "solved" by the determination of x to effect this arrangement.

In proceeding to a solution by physical means, it is but natural that we should simplify the process, first by removing the 12 loose things ; and then by separating the 108 remaining things into 3 equal subgroups, so that we shall have only a single square-group layer, comprising $108\div3=36$ things, to deal with.

Obviously, 36 things may be arranged to form a square group in 6 rows, with 6 things in each row ; and since the assets of a group may be increased either by the addition of actual asset things to the group, or by the withdrawal of liabilities, it is well that the double sign "$\pm$" be used before the numerical evaluation of x, giving the solution as

$$x=\pm6.$$

It should be carefully noticed, however, that this indicates merely the fact that if the 6 things forming the row are of a positive nature, the row must be repeated to 6 times; while if the 6 things forming the row are of a negative nature, 6 such rows must be withdrawn for compliance with the specified relation.

Cases do sometimes arise in practical calculations where the use of the double sign serves a useful purpose —for instance, when dealing with pressures on both sides of a piston, as in the case of Fig. 45; and in the bending of beams, distances to right and left of some convenient section being regarded as positive and negative because they exercise an influence upon the kind of bending produced by the loading—*i.e.* as to whether the centre of curvature is above or below the axis of the beam. Other instances will doubtless suggest themselves on consideration.

Taking next the slightly less obvious case:

$$5x^2+11=-234,$$

the statement evidently means that an assemblage, capable of arrangement in two subgroups—one consisting of 5 layers, each layer comprising x rows, with x things in each row; and the other comprising 11 loose asset things—has the net effect of a liability for 234 such things.

Obviously, then, the things in the 5 layers must represent a liability greater by 11 than 234—*i.e.* the $5x^2$ things must be *bills*, each demanding delivery of one thing.

In proceeding to a solution by physical means, it is but natural that we should simplify the task by first removing the 11 asset things, leaving the group comprising 245 bills (each demanding 1 thing), which are to be arranged in 5 similar square-group layers; and then by dividing the 245 bills into 5 equal subgroups, so that we have only one square-group layer, comprising $245 \div 5 = 49$ bills, to deal with.

Now, the 49 bills may be arranged to form a square group in 7 rows, with 7 bills in each row; and since the

assets of a group may be reduced either by the addition of bills or by the withdrawal of asset things, the solution may be denoted as

$$x = \{+(-7)\} \text{ and } \{-(+7)\};$$

or, more concisely, as

$$x = \{\pm(\mp 7)\}.$$

Whatever method of writing be adopted, the facts must be fully appreciated and plainly indicated—viz. that the 49 things represented as x^2 are *real* things; and the debt can be discharged only by the delivery of those things or some agreed equivalent. They are in no sense unreal or mysterious, for they could not be charged to the group as liabilities unless they had been delivered to that group at some time as assets—assuming (as we do in all mathematical investigations) a perfectly honest system of accountancy.

Adfected Quadratics.—The form of the quadratic arrangement which is partly of the second and partly of the first order (sometimes called the "adfected" quadratic) may be solved by the same straightforward means of simple arrangement as the pure quadratic; but the inclusion of the linear relation calls for some detailed consideration, while the type is of importance in that grave misconceptions regarding the true significance of the relation and its results have been introduced through failure to keep the physical realities clearly in view.

Let us consider a typical example:

$$3x^2 + 33x = -84.$$

Here the arrangement-specification indicates an assemblage capable of arrangement in two subgroups—the one consisting of 3 layers, each layer comprising x rows, with x things in each row; and the other comprising 33 rows, with x things in each row—while the total effect of the assemblage is a liability for 84 things.

Evidently it would be simpler to divide the assemblage into 3 equal parts, so that we have only one square-group layer to deal with, the statement becoming

$$x^2+11x=-28.$$

Further, it is obvious that if 28 loose things be added to the group, its effect will be zero—that is, neither assets nor liabilities—while the arrangement-specification will be $(x^2+11x+28)$; and clearly, some of the things in the groups dependent upon x must be bills.

To determine x, all we have to do is to arrange the assemblage in a rectangular group—*i.e.* to "factorise" $(x^2+11x+28)$—and this is the basis upon which rests the whole process of solving quadratic equations. Indeed, it will be seen presently that, although we dignify this branch of our "counting" by the special name of "quadratic equations," it is really neither more nor less than a method of factorisation for *any* expression of the type x^2+ax+b, whether factorisation in whole numbers is possible or not—provided always that the assemblage represented by the expression will submit to factorisation (see p. 46, and also later pages in this Chapter).

By the methods indicated in Chapter III. it is clear that an assemblage of $(x^2+11x+28)$ things may be arranged to form a rectangular group in $(x+7)$ rows, with $(x+4)$ things in each row, as indicated in Fig. 47; and since the 7 and the 4 are both positive, it is evident that x must be negative in both cases if the effect of the whole assemblage is to be zero.

The result of the factorisation indicates that the assemblage must comprise either:

(1) Assets of 4 things, and liabilities of x things, all repeated to $(x+7)$ times; or

(2) Assets of 7 things, and liabilities of x things, all repeated to $(x+4)$ times.

The effect of the assemblage may be zero—*i.e.* no assets and no liabilities—if either:

(*a*) The liabilities are 4 things, giving $x = -4$, and $(x+4) = 0$; or

(*b*) The liabilities are 7 things, giving $x = -7$, and $(x+7) = 0$.

Now, these evaluations of x must be admitted in the "repetition-factors," as well as in the asset-groups, for (as shown in Chapter II.) we have but to change our view-point to see as "rows" what formerly we saw as "things in each row," and *vice versa*; the actual assemblage

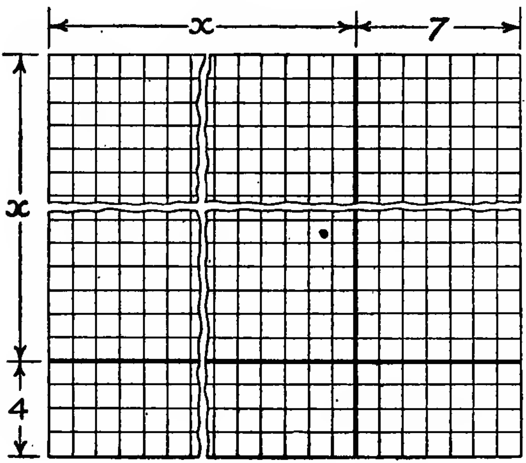

FIG. 47.

remaining totally unaffected by any such change of view-point.

Hence, if $x = -4$ in the statement (1) above, that statement as to the total contents of the assemblage will become either:

(1*a*) Assets of 4 things, and liabilities of 4 things, all repeated to $(-4+7)$ times, the -4 in the brackets denoting, of course, negative repetitions—*i.e.* *withdrawals*; or

(1*b*) Assets of 7 things, and liabilities of 4 things, all repeated to $(-4+4)$ times, the -4 in the brackets again denoting *withdrawals*.

Similarly, if $x=-7$ in the statement (2) above, that statement will become either:

(2*a*) Assets of 7 things, and liabilities of 7 things, all repeated to $(-7+4)$ times, the -7 in the brackets denoting *withdrawals*; or

(2*b*) Assets of 4 things, and liabilities of 7 things, all repeated to $(-7+7)$ times, the -7 in the brackets denoting *withdrawals*.

These four possibilities are covered by the two "solutions"

$$x=-4, \text{ and } x=-7,$$

provided that we take only one value for x at a time in particularising the assemblage (which is no more than reasonable, since x represents a number of things, and can, therefore, have but one value in relation to any particular group); and provided also that, in evaluating x^2 (*i.e.* $x \times x$), we faithfully represent the appropriate possibility—(1*a*), (1*b*), (2*a*) or (2*b*) above.

Thus, if $x=-4$, x^2 represents four *withdrawals* of four *bills*, or some other operation producing an equivalent effect; and if $x=-7$, x^2 represents seven *withdrawals* of seven *bills*.

Hence, there are two distinct assemblages which satisfy the equation

$$x^2+11x+28=0;$$

viz.

(1) when $x=-4$:

x^2 = four withdrawals of four bills;	
equivalent in effect to . . .	16 asset things
28 = merely	28 asset things
Total . . .	44 asset things
$11x$ = eleven repetitions of four bills;	
equivalent in effect to . .	44 liability things
Total effect . . .	0

and

(2) when $x = -7$:

x^2 = seven withdrawals of seven bills; equivalent in effect to . . .	49 asset things
28 = merely	28 asset things
Total . . .	77 asset things
$11x$ = eleven repetitions of seven bills; equivalent in effect to .	77 liability things
Total effect . . .	0

Both of these assemblages may be actually set up on a Halma board, using (say) green pieces to represent each an asset-thing, and (say) red pieces to represent each a liability-thing—*i.e.*. a thing which must be regarded as a debt or charge upon the group.

These two solutions, of course, both satisfy the given equation

$$3x^2 + 33x = -84.$$

Now let us consider another typical case:

$$x^2 + 7x = 18.$$

The arrangement-specification indicates that an assemblage which has the effect of 18 asset things (we shall see presently that the assemblage comprises asset things or vouchers to some greater number of things than 18, and also bills reducing the net effect to 18 asset things) is to be arranged in two subgroups—one a square group of x rows, with x things in each row; and the other comprising 7 rows, with x things in each row.

If we remove 18 things, the net effect of the assemblage will be zero—*i.e.* the assets will exactly balance the liabilities. Hence the suggestion above, that the group may comprise both assets and bills in addition to the 18 stated things; for, in the absence of knowledge concerning the relation of x to 18, we should have no justification for assuming that the square group of x^2 things and the 7 rows of x things would disappear entirely on our removing the 18 asset things from the group.

If the 18 things be removed, the arrangement-specification becomes

$$x^2+7x-18=0\ ;$$

and, by the methods described in Chapter III., the assemblage may be arranged in $(x+9)$ rows, with $(x-2)$ things in each row, giving

$$(x+9)(x-2)=0.$$

Applying the arguments, and complying with the stipulations, of the preceding example, this indicates the solution as

$$x=-9, \text{ and } x=+2.$$

The full significance of this should be developed by the student, on the lines indicated above in the interpretation of the preceding example.

Any other equation in which the assemblage, after treatment to make the assets exactly balance the liabilities, will submit to factorisation in whole numbers may be readily solved in the manner shown above.

Let us consider next a case in which the ordinary methods of factorisation fail—for instance :

$$x^2+12x+23=0.$$

We might, of course, attempt the factorisation physically by the method of trial and error ; but the following method, by means of logical argument readily capable of physical demonstration, is at once more simple and direct.

Imagine the x^2 group laid out as in Fig. 48, with the 12 rows of x things grouped some along each of two adjacent sides, leaving accommodation for some number of loose things at the right-hand bottom corner to complete the rectangular grouping.

The greatest number of loose things for which we can provide accommodation is, in this case, 36—obtained by placing 6 of our 12 rows of x along one side, and the other 6 rows along an adjacent side, of the x^2 group, as indicated in Fig. 49.

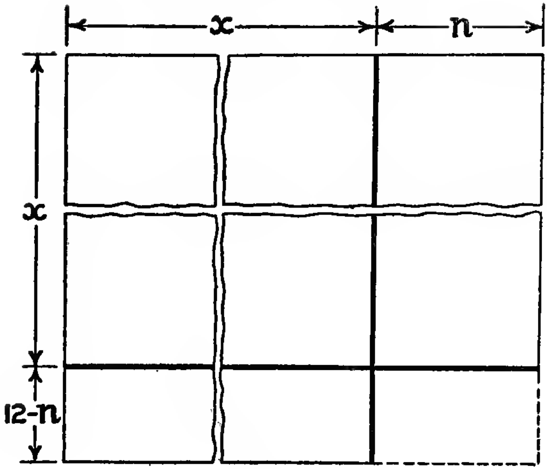

FIG. 48.

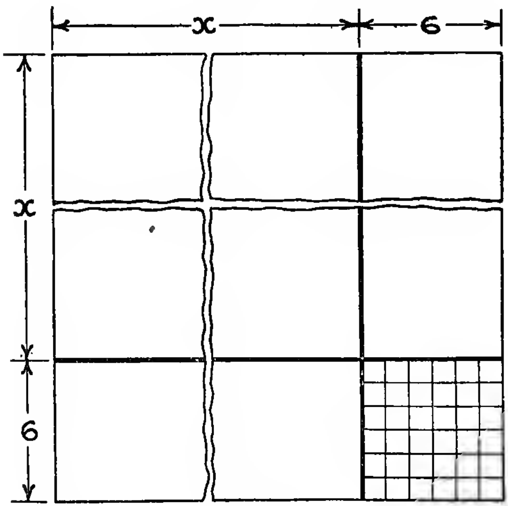

FIG. 49.

Whether all this accommodation be needed or not, it is well to start with it as a basis from which reductions may be allowed if the whole is not required. It is extremely important to notice that, *with the linear part of the relation comprising* 12 *rows of x things,* 36 *is the maximum number of things which can possibly be accommodated.*

Let us suppose for a moment that we have to accom-

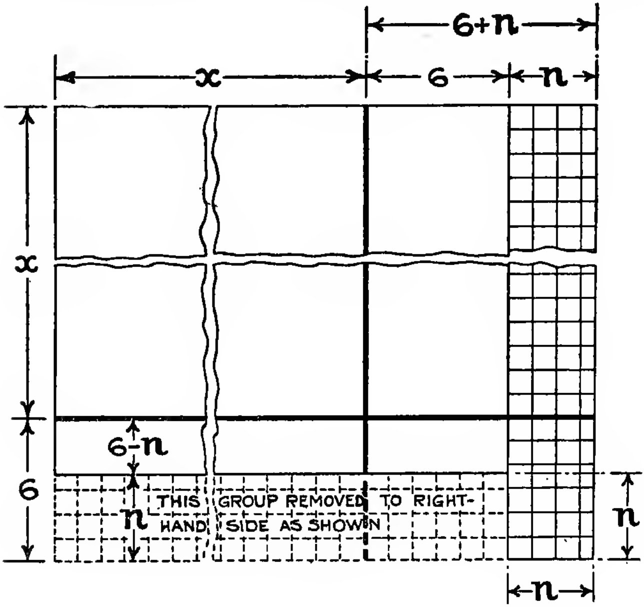

FIG. 50.

modate the full 36 things possible, as in Fig. 49 ; and then we can, as it were, crowd out the 13 things so accommodated in excess of the actual requirements. This crowding out may be effected by transferring some of the rows of $(x+6)$ things from the bottom to the right-hand side of the group, as indicated in Fig. 50 ; there being then $(6+n)$ rows along the right-hand side, and $(6-n)$ rows along the bottom.

Now, it will be seen from Fig. 50 that the process of "crowding out" has had the effect of marshalling the things crowded out into a square group, comprising n rows, with n things in each row; and hence, as we have to crowd out $36-23=13$ things,

$$n^2=13,$$

and

$$n=\sqrt{13}=3\cdot605.$$

This will be more significant if stated as

$$n=\sqrt{36-23}$$
$$=\sqrt{6^2-23},$$

and still more significant if stated as

$$n=\sqrt{\left(\frac{12}{2}\right)^2-23}.$$

It follows, then, that the assemblage of $(x^2+12x+23)$ things may be arranged in

$$\left[x+\left\{\frac{12}{2}+\sqrt{\left(\frac{12}{2}\right)^2-23}\right\}\right] \text{ rows,}$$

with

$$\left[x+\left\{\frac{12}{2}-\sqrt{\left(\frac{12}{2}\right)^2-23}\right\}\right] \text{ things in each row;}$$

and, if the net effect of the whole assemblage is to be zero, the solution may be stated as

$$x=-\frac{12}{2}\pm\sqrt{\left(\frac{12}{2}\right)^2-23}$$
$$=-6\pm\sqrt{13}=-6\pm3\cdot605$$
$$=-2\cdot395, \text{ and } -9\cdot605.$$

The most particular and careful notice should be given to our reason, as indicated above, for using the double sign "$\pm$" before the $\sqrt{13}$. If the argument be pursued faithfully, it will be seen that we do *not* use this sign in order to comply with the oft-repeated statement (which we have shown to be unjustifiable and false) that both

$(+3{\cdot}605)$ and $(-3{\cdot}605)$ if "multiplied by themselves produce 13." The 13 things may be arranged to form a square group in 3·605 rows, with 3·605 things in each row; and the nature of the things is not in any way affected or concerned in their arrangement thus. We use the double sign "$\pm$" merely to show that 3·605 of the rows of x things must be *taken from* one side of the x^2 group, and *added to* those lying along the adjacent side, as indicated

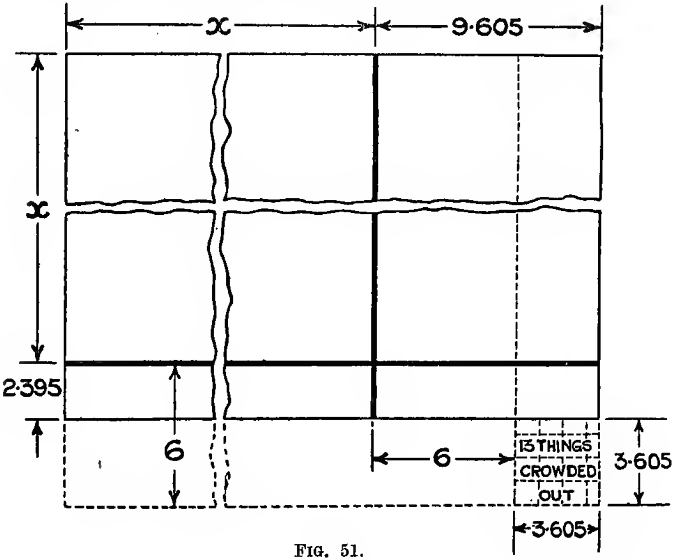

Fig. 51.

in Fig. 51, in order to "crowd out" the 13 loose things accommodated in excess of the specified number.

The student is strongly recommended to "solve" a fair number of typical quadratic equations on the lines indicated above, demonstrating each and every step in the reasoning by the actual arrangement and rearrangement of real things.

It will be seen that, taking the standard form of the quadratic as

$$x^2+bx+c=0$$

(and, clearly, any quadratic assemblage or group of things may, by the simple processes described in the preceding pages, be reduced to a form suitable for accounting or expression thus), the solution is

$$x=\left\{-\frac{b}{2}+\sqrt{\left(\frac{b}{2}\right)^2-c}\right\}, \text{ and } \left\{-\frac{b}{2}-\sqrt{\left(\frac{b}{2}\right)^2-c}\right\};$$

which, provided there be no vagueness or misconception regarding the use of the double sign, may be written

$$x=\left\{-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-c}\right\}.$$

As a fair example to show the practical application of this method for solving quadratics, consider the equation

$$x^2-5\cdot62x-4\cdot96=0.$$

By the arguments indicated above:

$$\begin{aligned} x &= +\frac{5\cdot62}{2}\pm\sqrt{\left(\frac{5\cdot62}{2}\right)^2+4\cdot96} \\ &= +2\cdot81\pm\sqrt{2\cdot81^2+4\cdot96} \\ &= +2\cdot81\pm\sqrt{7\cdot8961+4\cdot96} \\ &= +2\cdot81\pm\sqrt{12\cdot8561} \\ &= +2\cdot81\pm3\cdot5855 \\ &= (+6\cdot3955) \text{ and } (-0\cdot7755). \end{aligned}$$

In this, all the steps are shown fully set out; but it will be obvious that several of them could be quite well performed mentally in actual calculation.

Where the coefficient of x^2 is fractional, the student is advised to regard the "reduction" of the equation to the standard form (with the coefficient of x^2 unity) from the common-sense point of view, instead of bothering with awkward ideas of "dividing by a fraction." For instance, in an equation such as

$$\tfrac{1}{4}x^2-2\cdot35x-6\cdot2=0,$$

notice that we have only a quarter of an x^2 group, and the

process of solution will be simplified by increasing the whole assemblage sufficiently to include a complete x^2 group. This can be done by *repeating to four times* each of the subgroups, giving the relation

$$x^2 - 9{\cdot}4x - 24{\cdot}8 = 0\ ;$$

and clearly, the values of x which satisfy this equation will satisfy also the given equation.

" Imaginary " Quantities.—It is frequently stated that if, in the equation

$$ax^2 + bx + c = 0,$$

the product $4ac$ be greater than b^2, the roots are " imaginary," or " unreal."

Translating this into the terms of the foregoing argument, it would seem that, if c be greater than $\left(\frac{b}{2}\right)^2$ in the equation

$$x^2 + bx + c = 0,$$

the roots should become imaginary or unreal.

The reason for this supposition is that, in such conditions, we should be required to extract the square root of a negative quantity.

Now, we have already seen (p. 73) that the process of extracting a square root is merely one of the physical arrangement of real things ; and is in no way affected by or concerned with the nature or effect of the things under arrangement. Hence, the question as to whether the things denoted by the number in the square root sign are " positive " or " negative " could not, of itself, introduce any difficulty ; and it certainly could not be a sufficient reason for the " roots " of the relation undergoing so stupendous a change as the passage from reality to unreality—for real things forming an ordinary assemblage passing into a state in which they have no physical counterpart.

Let us see the real significance of this matter.

Take the equation

$$x^2 + 12x + 38 = 0,$$

the left-hand side of which assuredly represents a possible assemblage; but which, seeing that 38 exceeds $\left(\frac{12}{2}\right)^2 = 6^2 = 36$, leads to the inference of "imaginary" roots of the form

$$x = -6 \pm \sqrt{36-38} = -6 \pm \sqrt{-2}.$$

Examining the given statement as an "arrangement-specification," we see that the assemblage (which must evidently comprise both assets and liabilities to equal amounts) is to be arranged in three subgroups:

(1) A square-group comprising x rows, with x things in each row;

(2) 12 rows, with x things in each row; and

(3) 38 loose things.

From this consideration the fallacy is at once apparent, for *the assemblage cannot be arranged to form a rectangular group.* With the linear part of the relation comprising 12 rows of x things, the greatest possible accommodation at the right-hand bottom corner is for $\left(\frac{12}{2}\right)^2 = 6^2 = 36$ things only. Hence, the extra 2 things, by which the specified 38 exceeds the maximum of 36 for which accommodation can be provided, are merely *homeless.*

In writing the equation

$$x^2 + 12x + 38 = 0,$$

we have set ourselves the impossible task of housing 38 things in a space sufficient for only 36 things; and hence—as might have been expected—we have 2 things left over.

Now, if two men were shut out of an hotel because there were no beds available in which they might sleep, they would probably suffer more or less inconvenience and discomfort; but it is doubtful whether this could be properly regarded as sufficient to render them "imaginary," or "without physical counterpart"; and it would be dis-

tinctly unpleasant if the victims of such a misfortune were, as a direct consequence, transformed straightway into unreality.

In the solution $x=\{-6\pm\sqrt{-2}\}$, the " -2 " in the square root sign denotes simply the two things for which no accommodation can be provided in the rectangular group into which the assemblage must be arranged to effect its "factorisation"; and so long as the equation stands as stated, with 12 rows of x things as the middle term of the arrangement-specification, so long will 36 be the greatest number of loose things for which accommodation can be provided—no matter what x may be, as regards either the number or the nature of the things which it represents.

One more instance will suffice to show the truth of the argument.

Given the equation

$$x^2+10x+42=0,$$

the rule for solution would indicate the roots as

$$x=-5\pm\sqrt{25-42}=-5\pm\sqrt{-17};$$

and the " -17 " in the square root sign denotes merely the 17 things for which accommodation cannot be provided so long as there are only 10 rows of x things for disposal along two adjacent sides of the x^2 group.

Granted compliance with the terms and stipulations of the quadratic arrangement, as set out on p. 169 *et seq.*, an assemblage of $(x^2+10x+42)$ things, some assets and some liabilities, could never, under any circumstances, be of zero effect—*i.e.* for no value of x whatsoever could the assets just balance the liabilities. For if x be negative, it denotes (as shown above) not only *bills*, but also, in the x^2 group, *withdrawn bills*.

The least value which this particular assemblage could possibly have is 17 asset things; and those are the 17 things which cannot be accommodated in any rectangular grouping of the assemblage.

The student should consider this matter well, and satisfy himself thoroughly as to his full comprehension of the underlying realities. It may be thought that the 17 things in the case above might be disposed along the bottom, say, of the rectangular group into which the remainder of the assemblage may be arranged, as indicated in Fig. 52; but, clearly, this would be equivalent to

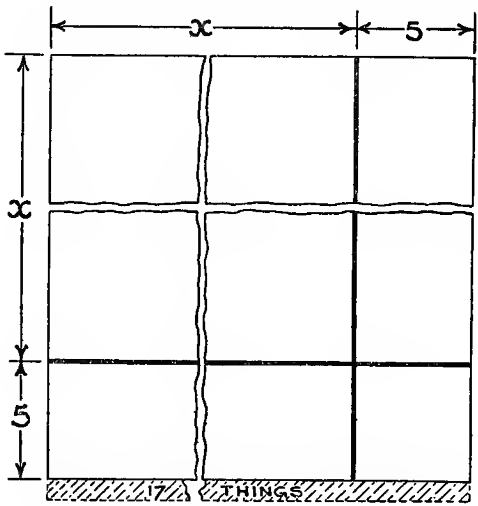

Fig. 52.

increasing the number of rows of x things—*i.e.* it would have the effect of an unauthorised alteration in the linear part of the relation.

It should be observed that an equation involving "imaginary" roots cannot occur in any calculations of practical engineering *unless the facts be falsely stated*; for the operations of such work do not either require or permit us to waste time and energy in endeavours to make a pint-pot hold a quart of liquid at one time. Hence, if such a case were to arise, the only proper course would be to immediately re-examine the facts and statements, so that

the fallacy or error may be located and rectified as soon as possible—for such there must be, beyond the possibility of doubt or question.

The conditions for "imaginary" roots can only arise when we simply write down a statement regarding an assemblage of things which cannot be realised. In real life, we are not justified in writing down an equation at will (or whim), regardless of the question as to whether it represents a practicable state or no, any more than we may coin a word without reference to actual things or experiences. If we do so, the results obtained will be merely fantastic—if not ridiculous; and in any case they are not likely to serve any useful purpose, while they may do a great deal of harm.

CHAPTER IX

EQUATIONS AND THEIR GRAPHS

Graphs generally.—It is not necessary here to discuss at length the extremely useful purposes which may be served by graphs apart from equations. The student is doubtless familiar with many ways in which graphs are plotted from day to day, enabling an engineer to keep, as it were, his finger on the pulse of a large and complicated undertaking. Many a leakage or lack of efficiency in some department of the work has been detected and remedied through such means; and if they are properly conceived and executed in relation to the particular purposes in view, graphs of this kind may be made enormously effective and helpful.

Sometimes it is of use to form an idea as to the relation according to which the two groups of things plotted on the squared paper are varying with respect to each other; and if the simplicity of the matter were more generally known and appreciated, such methods would undoubtedly be used far more extensively than they are. For a graph plotted upon numbers ascertained from day to day or from week to week must inevitably depart from the basic relation; while the human element gives rise to further departures in many different ways. These small disturbances may be more or less eliminated if the graph be skilfully interpreted, with discernment and imagination, on a judiciously broad view, and the basic relation particular-

ised accordingly—not only as to the type or form of the variation, but also as to the rates and constants.

For instance, in dealing with the output or productivity of a workshop in which some complete article is made in quantities, the figures obtained from the time-books, checkers' records, etc., will take no account of many highly important factors ; and the graph alone may either fail to give sufficient prominence to some slight falling off in efficiency, or—by reason of some peculiarly favourable circumstance acting in opposition, with regard, perhaps, to some totally different section or phase of the work—may even indicate a slight improvement instead of a loss. If the relations according to which the primary groups of things concerned vary with one another be determined with reasonable accuracy, much more comprehensive and specific information may be obtained ; and reliable forecasts regarding the probable effects of changes—proposed or to be expected—in the conditions of working may be formed without difficulty.

Care should be exercised to guard against the erroneous view (somewhat commonly held) that one has only to plot a few graphs and hang them conspicuously around the office to ensure perfect efficiency in the work of other people; and also the view—equally erroneous—that every kink and spike in a graph represents a correspondingly definite eruption in the actual work under record. A graph *can* lie—indeed, it is safe to assume that, unless it be interpreted with true discernment based upon an intimate and full knowledge of the facts, it will succeed in conveying false impressions at frequent intervals, no matter how precisely it be plotted.

The power to conceive graphs—to see which are the pairs of groups which need the most careful watching in their variations ; and to devise the most simple and effective methods for recording those variations in graphical form—is far more important than the ability to merely cover acres of squared paper with lines in different colours

and full of violent jerks. More important still is the power to interpret a wisely conceived and intelligently prepared graph—to visualise the realities clearly, instead of seeing only the diagrammatic representation of them.

In all practical engineering work, the variations of dependent groups are of a simple nature. Many follow the linear relation, while a large proportion of the remainder are quadratic; and in deducing an equation to correspond with a graph plotted from figures recorded on actual observations, the greatest care should be taken to adopt the type of relation which is consistent with the actual facts of the case, and to evaluate both the rates and constants faithfully in accordance with those facts. Moreover, where the human element is involved (and human nature still plays an important part in all engineering work), one must be prepared for subtle—but none the less definite and important—changes in the *rates* of variation themselves.

An instance of this from actual practice will show the kind of thing which happens, and may be not without interest to the student.

Articles, sufficiently similar for practical purposes, were being (and to be) produced in large numbers; and the obvious suggestion was made that a graph be plotted, showing the numbers of men employed (these being paid a practically uniform wage), together with a second graph showing the numbers of articles produced, week by week.

Each article having to pass through a fair number of stages in different hands, the slightest hitch in any single department inevitably slowed down the whole progress of output; and—more important still—the smallest lack of celerity in passing from stage to stage *during one day* was reflected in the output *for the week*. It was actually found, on closer observation, that the absence of a certain supervisor for a day or so could be relied upon to accompany an unwelcome change—*i.e.* a downward tendency—in the gradient of the output-graph when the figures for that week came to be plotted; and this, as was clearly proved,

was in no way due to retardation of the actual working in his department, but solely because he had devised means for getting the work into and out of his department very expeditiously. These means were so simple and efficient that nobody else was even aware of them; and hence, in his absence, the less expeditious methods employed in all other passages from department to department were employed. This discovery led to improvement in the methods of transmission, and a large saving was effected in consequence.

Now, as the number of articles produced per week increased with the expansion of the undertaking, the number of men employed naturally increased also. For some time there was difficulty in understanding why an influx of new men should produce (as it did repeatedly), not a gain, but actually a *loss* of output; and charges of inefficiency were laid against the administration on several occasions when this occurred. Of course, the explanation (when found) proved quite simple; and the administration showed at once as the victim instead of the culprit. There had been no proper looking ahead, with *gradual* increases in the numbers of men. New men had been started in considerable batches, and only when the pressure of work became so acute as to leave no alternative. Consequently, for some time prior to each influx of men, a very high state of efficiency had obtained; and this was upset by the new arrivals.

Not only were the new-comers comparatively unproductive for the first few days; but the process of working them up to the state of efficiency which had been reached before they came occupied some weeks, even with the best of them. Moreover, before each influx, all the men working were of proven suitability; whereas each new batch included some who, after a fair trial (lasting, perhaps, a few weeks) were found unsuitable, and had to be weeded out. And beyond all this, the efficiency of the old hands was adversely affected by the influx; for some had to help

and instruct the new-comers regarding special points of detail, while the operations of routine and supervision were inevitably disturbed to a considerable extent.

In addition to the eternal differences between individuals, natural processes will not submit to sudden startings or sudden stoppages. We can no more start ten men working at maximum efficiency upon a new task in a day or a week than we could start an engine working with high efficiency, loaded to its fullest capacity, in a few seconds.

Another point which soon attracted attention was the fact that the two separate graphs showed only *time-rates* of variation for the men and the articles; and, useful though they were, they could not show clearly the efficiency (or otherwise) of working—for efficiency (as regards the productivity of the men) was concerned solely with the numbers of men working in relation to the numbers of articles produced.

A new graph, showing the variations of these two groups directly with each other, was prepared; and valuable information was obtained by interpreting it properly, enabling the staff to be adjusted with efficiency, by gradual changes, ahead of the variations in the volume of work to be handled.

Variations in the actual rates of variation themselves form a highly important point, calling for careful attention.

For instance, in dealing with the efficiency-graph referred to above, suppose a series of plotted points showed on the diagram as indicated in Fig. 53.

One method, commonly employed, is to simply link up the points, as plotted, with straight lines—and this is the least instructive of all methods.

Another method is to draw the "best straight line" —*i.e.* the straight line which lies most evenly among the plotted points—as indicated in Fig. 53. This is better, for it shows the main relation between men and output as of the "linear" order—which is the only reasonable view on a broad basis. It fails, however, to take account of

variations in the working efficiency, and is thus incapable of indicating periods of increasing efficiency as against periods of decreasing efficiency, on the detection of which so much may depend.

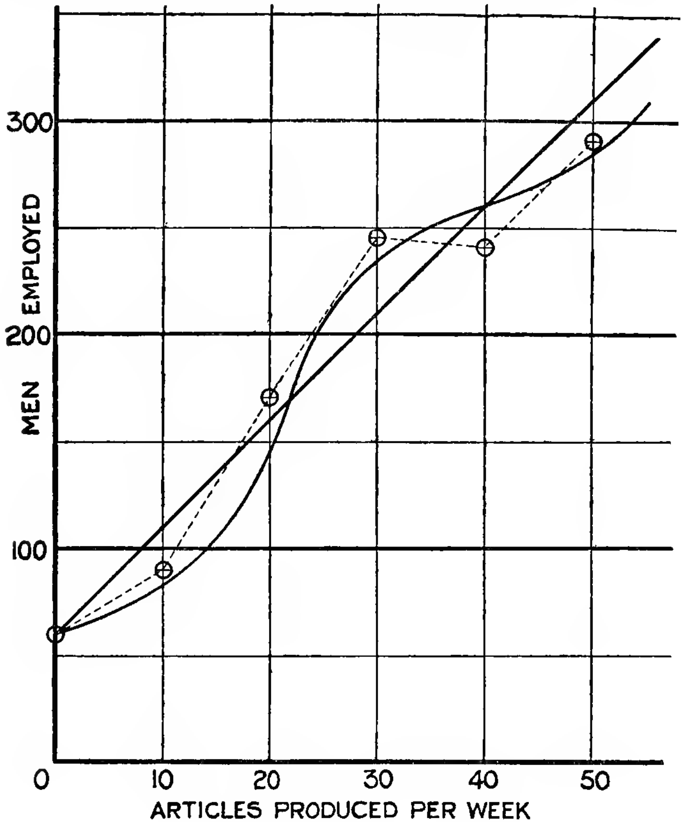

FIG. 53.

The strong point of this method lies in the fact that it shows the broad relation according to which, under the known conditions, output varied with the number of men employed, eliminating the differences between the produc-

tivities of different men and the disturbing influences of specially favourable or adverse temporary conditions. It should, therefore, be used in addition to the third method, described below; but great care and discernment are necessary in determining the *best* straight line.

Let it be always remembered that calculations and graphs are merely the rough "sighting-shots" of the real engineer, instead of being (as is erroneously supposed by many) his precise and rigid determinations. Calculations of all kinds are based partly upon more or less indefinite information, and partly upon mere assumptions; and the only proper light in which they can be regarded—and surely it is a sufficiently high ideal—is as the foundation or groundwork upon which the skill of the engineer (be he designer or administrator or both) may work with reasonable confidence.

The third method is to draw the curved line which the plotted points indicate, apart from more or less unavoidable errors in observation and recording, as connecting the numbers of men with their total productivities throughout the whole range of operations, instead of proceeding with abrupt jerks from total-point to total-point. The manner in which such a curve indicates periods and phases making for higher efficiency, as compared with those which produce the opposite tendency, will be seen from Fig. 53; and the inferences to be drawn are obvious.

It should, of course, be remembered that the basic relation between the numbers of men employed and the numbers of articles produced is of the linear order always. The undulations in the curve reflect merely changes in the *rates* of productivity. Thus, if the number of men employed be denoted by M, and the number of articles produced by P, the basic relation must be of the type

$$M = aP + b;$$

in which b denotes the number of men which must be employed to keep the organisation *ready* to produce,

without actually producing any articles at all; while a denotes the reciprocal of the average rate of productivity per man—*e.g.* if, on the average, each man produced 5 articles per week, a would be $\frac{1}{5}$; whereas, if 5 men were

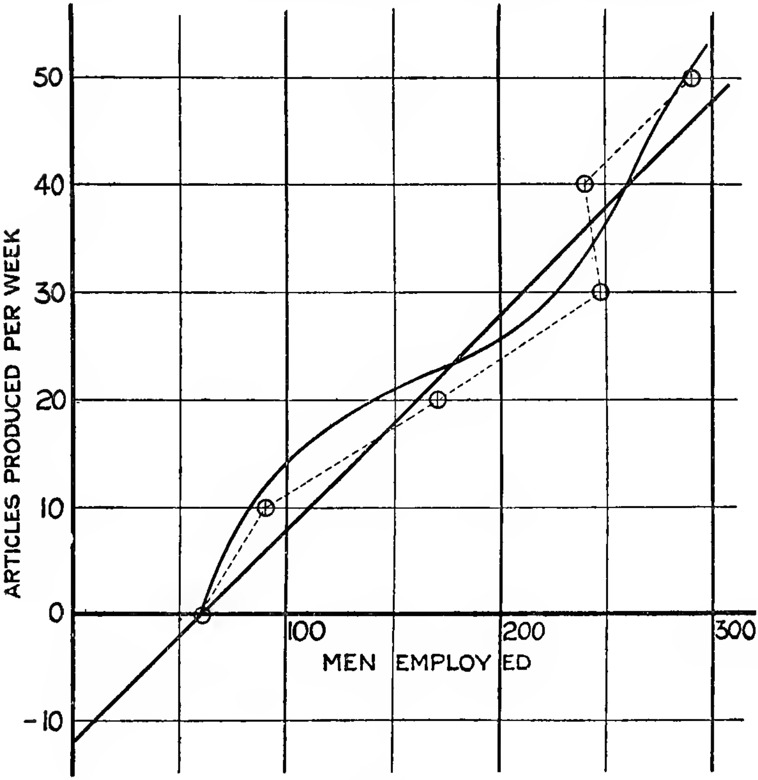

FIG. 54.

necessary to the production of 1 article per week, a would be 5.

Alternatively, P might be plotted vertically, and M horizontally, as in Fig. 54, when the broad relation would be

$$P = cM - d,$$

where c would denote the actual average rate of productivity per man, and d would denote the number of articles equivalent (as regards productivity) to the keeping of the organisation in a state of *readiness* to produce, without actually producing any articles—showing, of course, that there must be some number of men employed to give $P = 0$.

In both cases—but perhaps more plainly in the latter—the undulations in the curve show fluctuations in the productivity of the men, and, therefore, in the efficiency of the organisation as regards working for output. Variations in wages, working costs and selling prices, should, of course, be similarly watched, and adjusted with equal care.

Doubtless it will be clear that, by transposing and rearranging the first of the above two equations,

$$aP = M - b\ ;$$

whence

$$P = M\left(\frac{1}{a}\right) - \left(\frac{b}{a}\right);$$

from which it follows that the coefficients of the second equation are related with those of the first, c being equal to $\left(\frac{1}{a}\right)$, and d to $\left(\frac{b}{a}\right)$. The student should satisfy himself as to the physical reality of such a change, as well as to its permissibility and true significance.

As a means to the effective use and interpretation of graphs, an intimate knowledge of the forms which correspond to the various standard types of relations is clearly important—if not actually indispensable; and we will therefore briefly consider these, examining the effects of differences in the rates and constants.

Since we have discussed a number of typical equations somewhat fully in Chapters VII. and VIII., it will perhaps be well to use some of them, in addition to others, as a basis for our consideration of graphs in particular. The discussions regarding these equations in Chapters VII. and VIII. should be re-read carefully in conjunction with the following considerations of their graphs.

Graphs of the Linear Relation.—A graph corresponding to the relation $y=4x+9$ between $x=-2$ and $x=+4$, is shown in Fig. 55.

This does not call for special comment, except to emphasise the fact that the coefficient of x (*i.e.* 4) shows as the *rate of variation*, the ratio of "rise to going" being 4 : 1 ; while the term independent of x (*i.e.* 9) shows as the

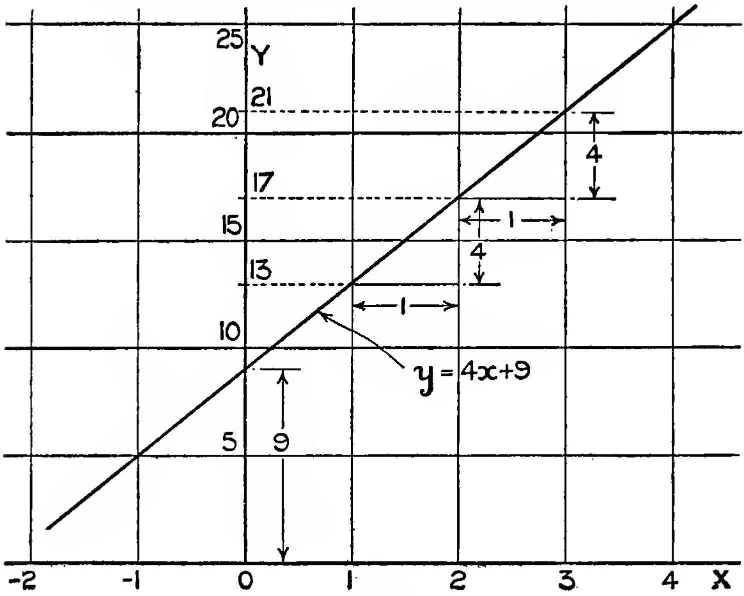

FIG. 55.

prop-height of the graph with relation to the point (sometimes called the "origin," and usually denoted by O) in which the rectangular axes OX and OY intersect.

A series of six graphs, corresponding to the typical forms of the Standard Linear Relation, is shown in Fig. 56 ; and from this it will be seen that, with the ordinary conventional method of plotting (*i.e.*, for x, positive values are plotted horizontally to the right, and negative values to

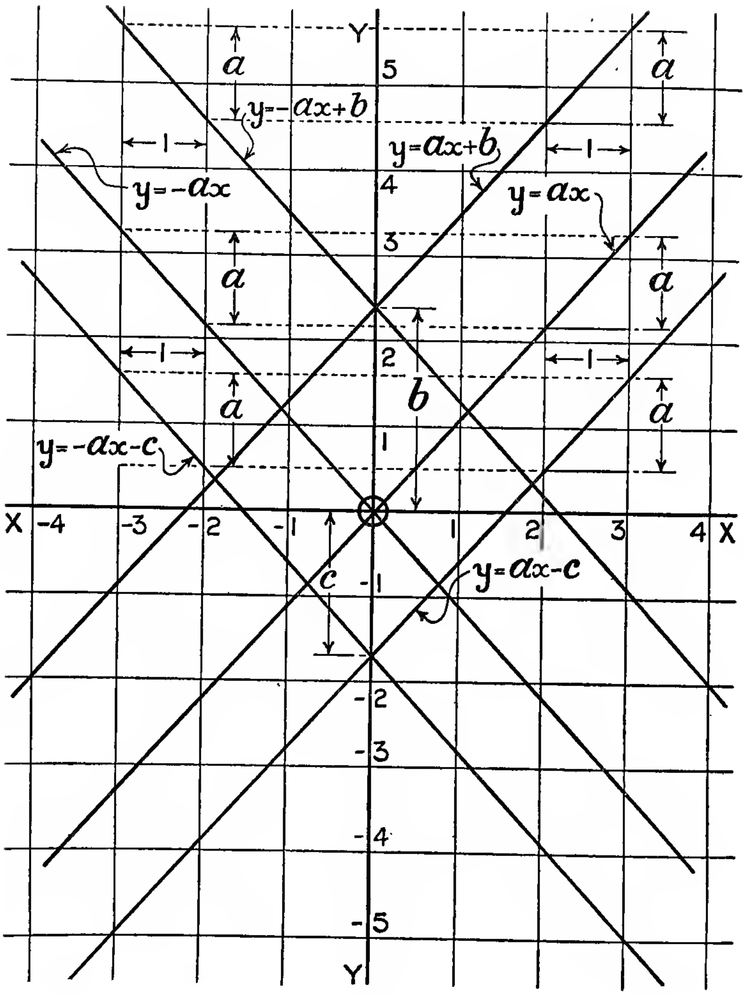
Y
5
4
3
2
1
-1
-2
-3
-4
-5
Y
X
-4
-3
-2
-1
1
2
3
4
X
a
a
a
a
a
a
1
1
1
1
b
c
y=-ax+b
y=ax+b
y=-ax
y=ax
y=-ax-c
y=ax-c

FIG. 56.

the left, of the origin; while, for y, positive values are plotted vertically above, and negative values below, the origin), the following inferences may be drawn:

(1) A graph which rises to the right corresponds with a relation in which the rate of variation is positive —*i.e.* the variation is direct;

(2) A graph which falls to the right corresponds with a relation in which the rate of variation is negative—*i.e.* the variation is inverse;

(3) The ratio of "rise to going" of a graph is equal to the coefficient of x (*i.e.* the rate of variation) in the algebraical statement of the relation, a fall being regarded as a negative rise;

(4) A graph which crosses the Y-axis above the origin (and which may therefore be thought of as "propped") indicates that the term independent of x in the relation is positive, and the "prop-height" of the graph is equal to that term;

(5) A graph which crosses the Y-axis below the origin (and which may therefore be thought of as "suspended") indicates that the term independent of x in the relation is negative, and the "suspension-depth" of the graph is equal to that term; and

(6) A graph which passes through the origin indicates that the relation contains no term independent of x—or, more properly, that such term is of zero value.

From these inferences, the particularisation of the linear relation to correspond with any given graph becomes a simple matter.

Thus, given the graph of Fig. 57, with the co-ordinates of two points on it, as shown, the "going" is $143-67=76$, and the corresponding "rise" is $39-16=23$, giving

$$a=23\div 76=0{\cdot}3026.$$

Also, in "going" to the left 76, the graph falls 23;

and hence, in going 143, it will fall $23\left(\frac{143}{76}\right)=43\cdot28$; leaving the "suspension depth" $39-43\cdot28=-4\cdot28$.

Hence, the relation which corresponds with the graph is

$$y=0\cdot3026x-4\cdot28.$$

Again, with the graph of Fig. 58, the "going" is

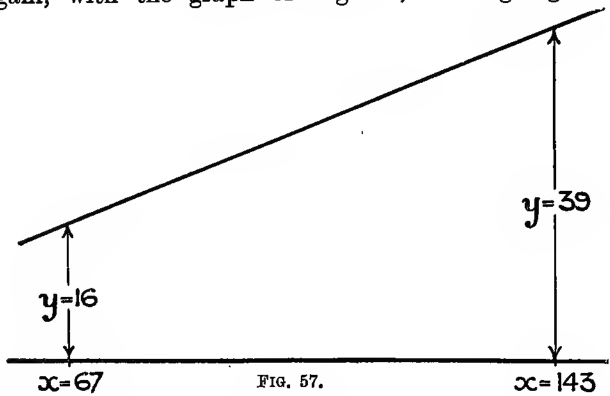

FIG. 57.

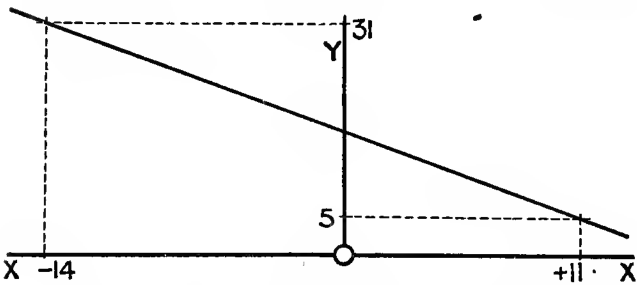

FIG. 58.

$14+11=25$, and the corresponding fall is $31-5=26$, giving

$$a=-26\div25=-1\cdot04.$$

Also, in going 25, the line falls 26; and hence, in going 14, it falls $26\left(\frac{14}{25}\right)=14\cdot56$; leaving the "prop-height" $31-14\cdot56=+16\cdot44$.

Hence, the relation which corresponds with the graph is

$$y = -1{\cdot}04x + 16{\cdot}44.$$

The Linear Relation is sometimes stated as

$$y = x \,.\, \tan\theta + b\,;$$

where θ is the angle between the graph and the horizontal (XOX) axis, parallel with which the values of x are plotted.

Care is necessary in using this form of the relation; for it is only true when both x and y are plotted to the same scale. For example, in the graph of Fig. 55, as drawn, the angle between the graph and the horizontal axis has not really a tangent equal to 4; and to use the same scale vertically and horizontally in such a case would be to render the diagram awkward and unwieldy without any compensating advantages being gained.

It is better, for most practical purposes, to think of the coefficient of x (*i.e.* the rate of variation) as represented by the ratio of "rise to going" in the graph; though there are cases in which the form $y = x \,.\, \tan\theta + b$ is convenient. One objection to it lies in its propensity for distracting attention from the important fact that the coefficient of x actually represents a *rate of variation* between the two groups —*i.e.* so many "y" things per unit variation in the "x" things—a fact which cannot be too clearly and constantly borne in mind.

In the same way, it will be seen from examination of Fig. 56 that if—and only if—the same scale be used both horizontally and vertically, the two graphs representing

$$y = +ax + b$$

$$\text{and } y = -\frac{1}{a}x + b$$

would be at right angles.

The discussion relating to Simultaneous Linear Relations (in Chapter VII.) should be re-read, and carefully examined, by the aid of graphs.

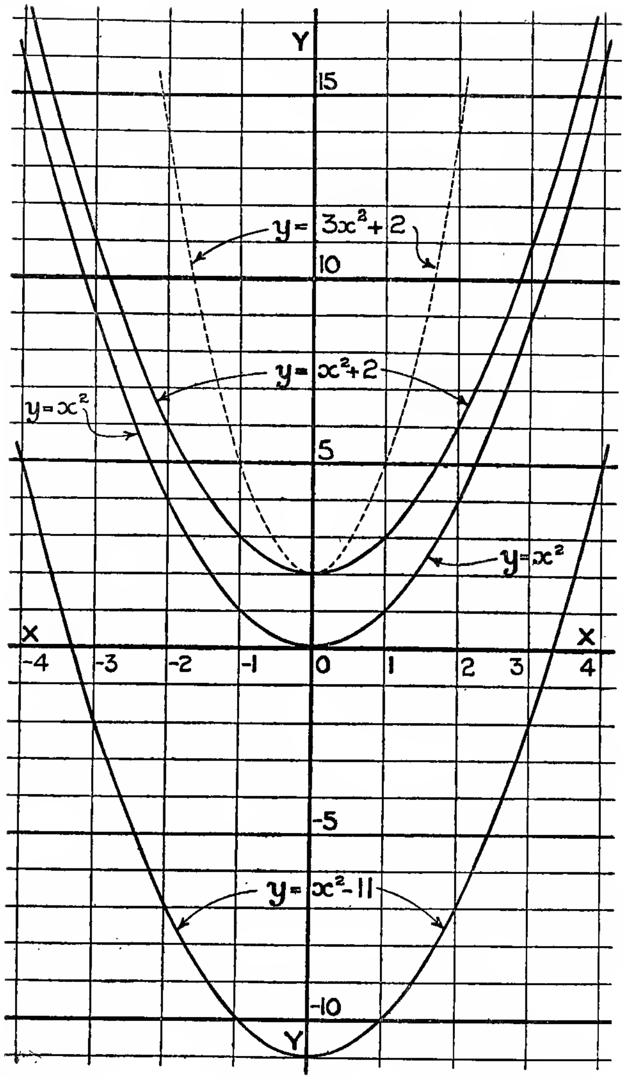

FIG. 59.

Graphs of the Quadratic Relation.—Three graphs corresponding to the relations

$$y = x^2,$$
$$y = x^2 + b,$$
$$\text{and } y = x^2 - c$$

are shown in Fig. 59; and from this it will be clear that the term independent of x is just a "prop-height-indicator," or "suspension-depth-indicator," as in the linear relation, its effect being equivalent to the lowering (if the term be positive) or raising (if the term be negative) of the X-axis in the diagram.

If there be a coefficient operating upon x^2, making the relation

$$y = ax^2 + b,$$

the curve is narrowed when a is more than 1 (as shown dotted in Fig. 59), and broadened when a lies between 0 and 1.

With the relation

$$y = -ax^2$$

the graph is as indicated in Fig. 60; and it should be observed that this corresponds with the "pure quadratic" $x^2 = -c$, the discussion of which (in Chapter VIII.) should be re-read in this light.

In Fig. 61 two graphs are shown, corresponding with the standard form of the quadratic relation.

The left-hand graph represents

$$y = x^2 + 6x + 8;$$

and the right-hand graph represents

$$y = x^2 - 4x - 11.$$

The central graph, representing

$$y = x^2 + 8,$$

is shown for comparison with the other two, since there are several points of interest and importance to be noticed.

First, the student should satisfy himself that all three graphs are exactly similar as regards *shape*, the differences between them consisting solely in their positions with relation to the origin. This may be done by plotting them afresh, carefully and to a fairly large scale, taking a tracing

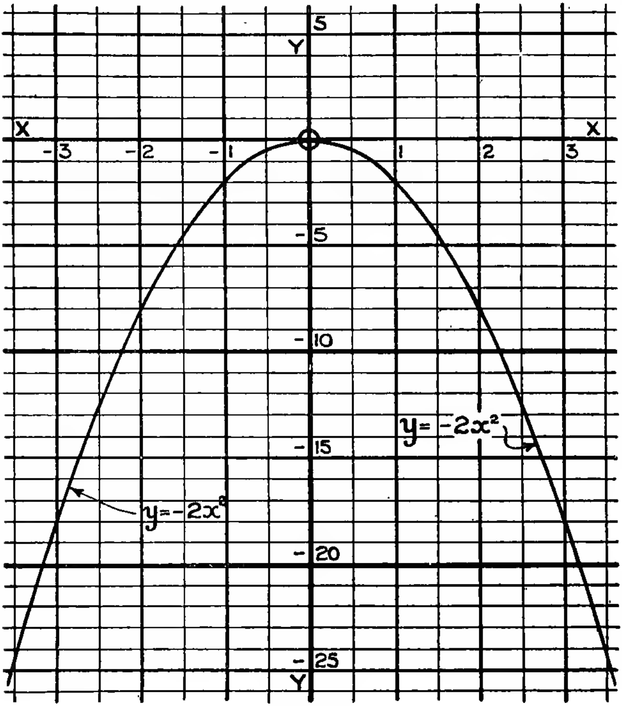

FIG. 60.

from one of the curves, and applying it in turn to the other two. It will be found a fact; and so long as the coefficient of x^2 remains unchanged, the *shape* of the graph is not altered by the introduction of either the linear relation or a term independent of x. These introductions affect only the *position* of the graph with relation to the origin,

the latter simply raising or lowering the curve vertically, without causing any horizontal displacement, and the

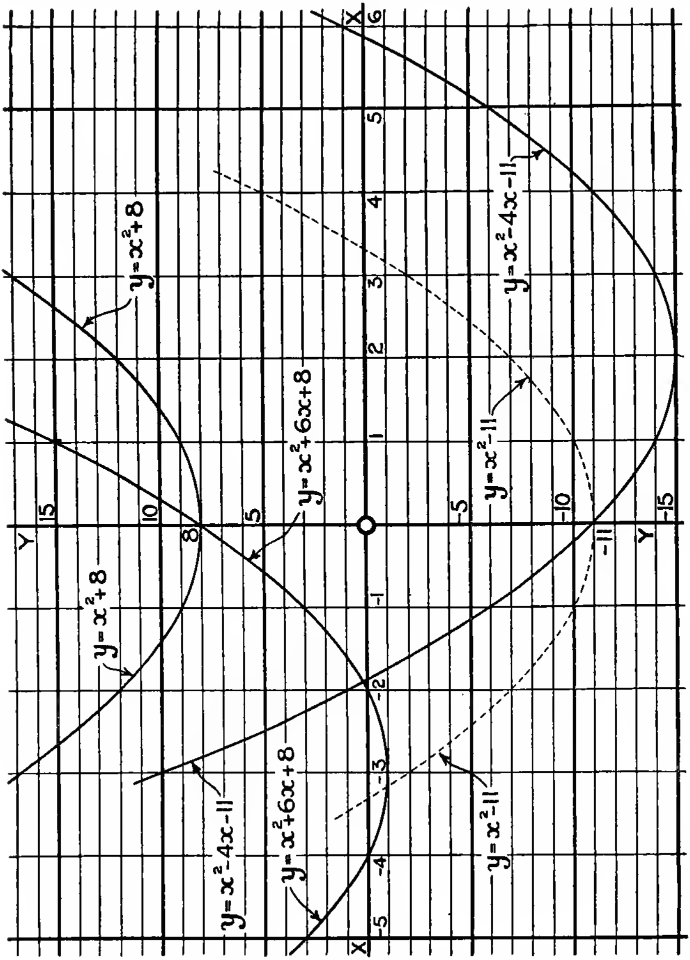

FIG. 61.

former causing both vertical and horizontal displacements, though the primary effect is a horizontal movement.

Considering first the left-hand graph, it will be seen that its prop-height is 8, exactly as is that of the central graph; but the axis of the curve has been moved 3 to the *left*, solely owing to the introduction of the term $(+6x)$.

Imagine the "prop" as a physical prop, 8 in height, and of some sensibly rigid material, standing on the origin. At the head of the prop, imagine a tiny wheel mounted upon a horizontal spindle, so that the wheel can rotate freely in the plane of the graph; and let this wheel have a circumferential groove in which the central graph (imagined as a templet of thin wood or sheet metal) can slide, the axis of the graph-templet being maintained always vertical.

When the graph-templet has slidden (remaining always in contact with the wheel at the prop-head) until its axis of symmetry has moved 3 to the left of the origin, the prop-head will be in contact with that point on the curve which formerly stood above $x=3$. The height of this point was formerly 17, and is now 8; which means that the curve has been lowered bodily by 9 (*i.e.* by $6^2 \div 4 \times 1$)—as a consequence of its movement towards the left—through the introduction of the term $(+6x)$.

Turning now to the right-hand graph, it will be seen that, for comparison with the central graph on a similar basis, the prop-height of the latter (*i.e.* 8) must be changed into a suspension-depth of 11—which means a lowering of the graph-templet bodily through a vertical distance of $8+11=19$, as indicated by the dotted graph in Fig. 61.

The introduction of the term $(-4x)$ has had the effect of moving the axis of the curve 2 to the right of the origin; and, again imagining the graph-templet as sliding in the circumferential groove of a tiny wheel at the lower extremity of the sling (or suspender), the point on the graph in contact with the wheel in the new position—*i.e.* the right-hand graph in Fig. 61—will be that which formerly corresponded with $x=-2$, its ordinate being -7. This point, then—

and with it the whole graph—has fallen through a distance equal to $11 - 7 = 4$, which is really $4^2 \div 4 \times 1$.

As a highly instructive exercise, the student is advised to draw and analyse the graphs corresponding to several sets of similar relations, such as $y = 3x^2 + 7x - 2$, and $y = 3x^2 - 5x + 4$; comparing them, on the lines indicated above, with a central graph representing $y = 3x^2$.

We may here anticipate later work slightly to observe that, given the relation

$$y = ax^2 + bx + c,$$

the differential coefficient (that is, the rate of variation) of y with respect to x is

$$\frac{dy}{dx} = 2ax + b.$$

At the lowest point of the curve (*i.e.* the point which, in the cross-section of a culvert or sewer, would be called the " invert ") $\frac{dy}{dx} = 0$ for obvious reasons ; whence

$$2ax_0 = -b,$$

and $$x_0 = -\left(\frac{b}{2a}\right).$$

This is the distance through which the basic curve, representing $y = ax^2 + c$, will have been moved horizontally owing to the introduction of the term (bx)—the movement being to the *left* of the origin if b is *positive*, and to the *right* of the origin if b is *negative*.

In either case, the curve will have been lowered bodily through a distance equal to

$$c - (ax_0^2 + bx_0 + c) = \left(\frac{b^2}{4a}\right),$$

its axis of symmetry remaining vertical throughout the movement, in consequence of its movement horizontally subject to the graph being always in contact with the head of the " prop " or the foot of the " sling."

These inferences are often useful in the practical interpretation of a graph.

For example, if a series of points, plotted from the records of actual observations, lies in some such form as is indicated in Fig. 62, and the facts of the case suggest a relation of the type

$$y = ax^2 + bx + c$$

between the two groups under consideration, the relation may be readily particularised by the following procedure.

Adjust the values of the co-ordinates to correspond

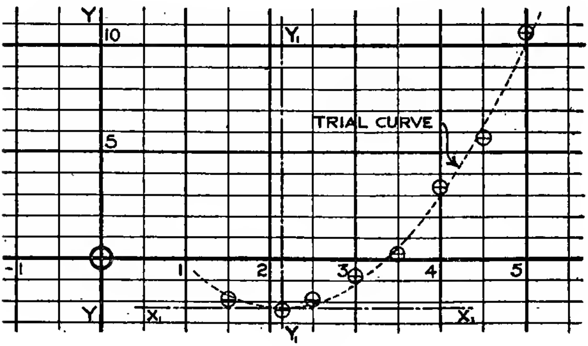

FIG. 62.

(vertically) with the axis of symmetry of the curve, and (horizontally) with the tangent at the invert, as indicated by $Y_1O_1Y_1$ and $X_1O_1X_1$ (respectively) in Fig. 62. The plotted points will then represent, with reference to these new axes, the curve which corresponds with a relation of the type

$$y_1 = ax_1^2.$$

Take the logarithms of the co-ordinates ; thus

$$\text{Log } y_1 = \text{Log } a + 2 \text{ Log } x_1.$$

Then, on a new diagram, plot the logarithms of y_1 vertically, and the logarithms of x_1 horizontally ; and

through these plotted points draw the "best straight line" *which has a ratio of rise to going* $=2:1$. If this straight-line graph has a "prop-height," its value gives log a (and in such case, a will be a positive number greater than 1), from which, of course, a may be at once determined. If the straight-line graph has a "suspension-depth," its value gives log a (and in such case, a will be a positive number between 0 and 1, *i.e.* a proper fraction), from which a may be readily determined.

The horizontal distance between the axes YOY and $Y_1O_1Y_1$ (Fig. 62) is equal to $\left(-\frac{b}{2a}\right)$; from which, a being known, b may be determined.

Applying the ascertained values of a and b to some known value of x, a value for (ax^2+bx) will be obtained; and the difference between this and the observed value of y which corresponds with the chosen value of x will give c. Several values of c should be determined thus from pairs of corresponding values for x and y, and the average taken.

The graph corresponding with the particularised relation may then be plotted and drawn; and its agreement (or lack of agreement) with the points plotted from observation will be at once apparent.

If this ascertained graph lie uniformly above or below the points plotted from observation, the facts of the case should be again examined, to see whether the values of b or c (or both of them) may properly be adjusted to give reasonable agreement between the particularised relation and the points plotted from observation. Bearing in mind the fact that there are probably errors and disturbances in the observed results, if the ascertained graph lies evenly among the plotted points—even though none of them lie actually on it—the particularised relation may be used (with discretion, of course) over a reasonable range of operation.

If the ascertained graph be plainly narrower or broader than the curve indicated by the observed results, the

straight-line graph of the logarithms of x_1 and y_1 should be re-examined, to see whether the "best straight line" chosen is really the best. With the graph narrower, perhaps a somewhat steeper straight line will give better agreement between the "law" and the observations; while if the graph be broader, a slightly less steep straight line may be preferable.

Should the straight-line graph of the logarithms be adjusted in this way, the relation will be of the type

$$y = ax^n + bx + c;$$

and the ratio of rise to going for the best straight line will be the value of n. A change in the slope of this straight line may or may not (but usually will) alter the value of a also, for obvious reasons.

Care is necessary, in all adjustments of this nature, to keep the facts and realities of the case clearly in view throughout; and to make only such alterations in the symbolism as are properly permissible and consistent with those facts and realities. Such cases seldom occur in practice, however.

The student should be careful always to avoid pressing a relation so particularised beyond the range over which it is properly applicable. Many errors have been made through failure to grasp the significance of this point. Because the "law" of, for example, a certain lifting tackle indicates that a pull of 20 lbs. (plus 10 lbs. to overcome frictional resistances) on the rope will raise a load of 1 ton, and that a pull of 40 lbs. (plus 10 lbs. to overcome frictional resistances) will raise a load of 2 tons, it does not follow that, with the same tackle, a pull of 2010 lbs. will raise a load of 100 tons.

Methods for Plotting Graphs.—A graph corresponding with a relation of the type

$$y = x^2 + bx + c$$

may be plotted in two portions: (1) a curve representing $y_1 = x^2$; and (2) a straight line representing $y_1 = bx + c$.

The ordinates of these two graphs may then be added, to give the ordinates for the single graph representing the complete relation.

This method is often useful for such cases as

$$ax^2 - bx - c = 0;$$

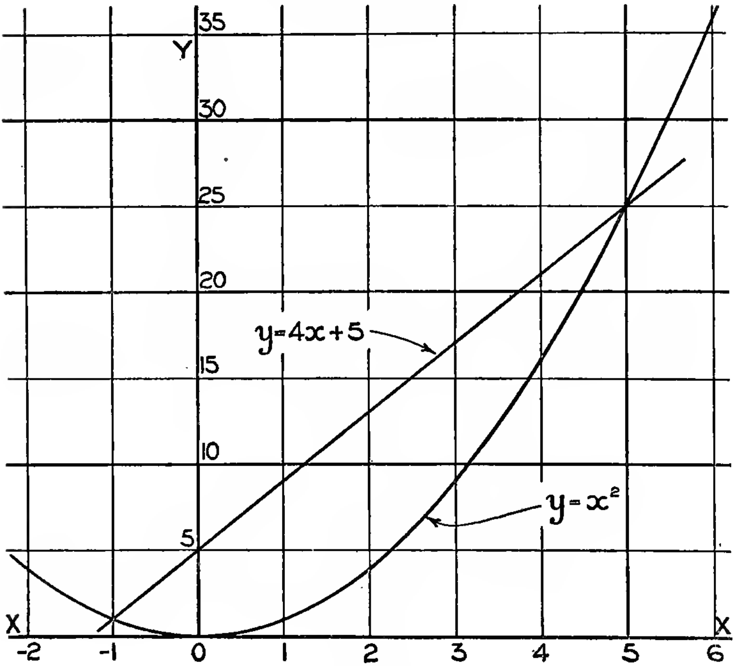

FIG. 63.

for the graphs of (1) $y_1 = ax^2$, and (2) $y_2 = bx + c$, will intersect in the points for which $y_1 - y_2 = 0$, showing the roots of the given equation with more precision than would the single curve, as indicated in Fig. 63, which represents the relation

$$y = x^2 - 4x - 5,$$

and shows the roots of the equation

$$x^2 - 4x - 5 = 0.$$

Much useful information may be obtained from a careful consideration of these points in relation to actual graphs; and the student is advised to examine for himself all the cases treated here (as well as others) on these lines.

The method may be extended and modified to suit relations of higher orders, and also those in which fractional indices of the independent variable render calculations for a single graph somewhat complicated.

For most cases, however, where a fairly extensive range of a graph is required rather than the mere roots of an equation, it is better to tabulate the calculations for the single graph.

Thus, for the relation

$$y = x^2 - 4x - 11,$$

the graph of which is illustrated in Fig. 61, the work would be set out as follows:

x	x^2	$4x$	$(x^2 - 4x)$	y
-3	$+9$	-12	$+21$	$+10$
-2	$+4$	-8	$+12$	$+1$
-1	$+1$	-4	$+5$	-6
0	0	0	0	-11
$+1$	$+1$	$+4$	-3	-14
$+2$	$+4$	$+8$	-4	-15
$+3$	$+9$	$+12$	-3	-14
$+4$	$+16$	$+16$	0	-11
$+5$	$+25$	$+20$	$+5$	-6
$+6$	$+36$	$+24$	$+12$	$+1$

Graphs of Higher Orders.—The need for treatment of graphs representing relations of orders higher than the quadratic seldom arises in practical work.

Such graphs are, however, sometimes useful in solving equations which might otherwise be troublesome, if not impracticable; and they are worth study for this reason alone. Moreover, a deal of information may be obtained with regard to the variations in a group of things represented by an equation if the graph of the relation be

plotted over a fair range; and the work of the Calculus is considerably simplified by the possession of a sound and intimate knowledge of graphs and their real meanings.

In Fig. 64 is shown the graph corresponding with the relation

$$y = x^3 - 6x^2 + 11x - 6,$$

indicating the three solutions of the cubic equation

$$x^3 - 6x^2 + 11x - 6 = 0.$$

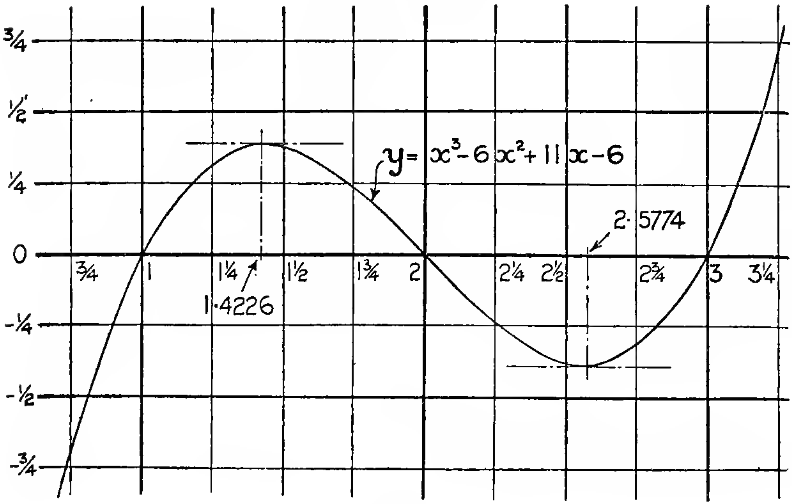

FIG. 64.

This arrangement-specification was chosen, for the sake of simplicity and clearness in illustration, because it may be factorised into $(x-1)(x-2)(x-3)$. The student will do well to investigate this graph; plotting it for himself, both as a single curve, and also by first plotting the three component graphs

$$y_1 = x^3,$$
$$y_2 = -6x^2,$$
and $y_3 = 11x - 6$;

afterwards adding the ordinates to obtain the ordinates for the complete graph of the given relation.

It will be observed that one effect of the term $(-6x^2)$ has been to move the axis of the basic curve to the *right* of the origin through a distance equal to the quotient (coefficient of x^2) $\div$ (index of x^3) $= 6 \div 3 = 2$. Other effects and influences may be seen, and traced to the terms causing them, upon consideration—especially in the light of the component graphs when plotted separately.

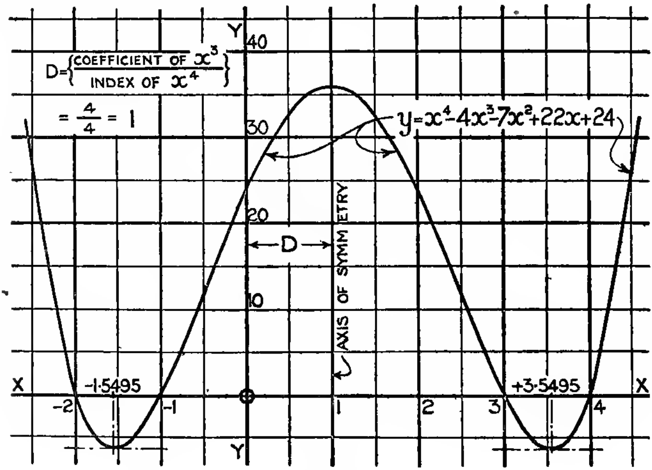

FIG. 65.

Other graphs of the third order should be plotted, and the conditions which render two of the roots "imaginary" should be carefully observed. It will be found that these conditions are exactly similar to those producing "imaginary" roots in quadratic equations—*i.e.* an attempt to accommodate more things than the relation provides for—and that the things which are sometimes regarded as imaginary are, in fact, merely *homeless*.

The same conditions apply to equations of all orders regarding some of their roots; but an equation of an *odd*

order (*i.e.* equations in which y is a function of x^1, x^3, x^5, ... x^{2n+1}, where n is an integer or zero) will always have at least *one real root.*

The graph corresponding to the relation

$$y = x^4 - 4x^3 - 7x^2 + 22x + 24$$

is shown in Fig. 65, over a range sufficiently wide to indicate the four roots of the equation

$$x^4 - 4x^3 - 7x^2 + 22x + 24 = 0.$$

Here, again, the arrangement-specification was chosen for the simplicity and clearness obtainable from the fact that it may be factorised into $(x+1)(x+2)(x-3)(x-4)$.

This should be carefully examined and analysed by the student, on the lines indicated for preceding cases; and graphs of other relations, of the fourth and higher orders, as well as some with intermediate fractional indices, should be plotted and drawn for similar purposes.

CHAPTER X

TRIGONOMETRY

Angles as Groups of Real Things.—There is often some difficulty in appreciating the fact that an angle, for the purposes of mathematical measurement and comparison, is treated as a group of real things ; and that trigonometrical calculations—like the rest of mathematical work—are concerned solely with the counting of real things in groups. This difficulty is unquestionably due to the common method

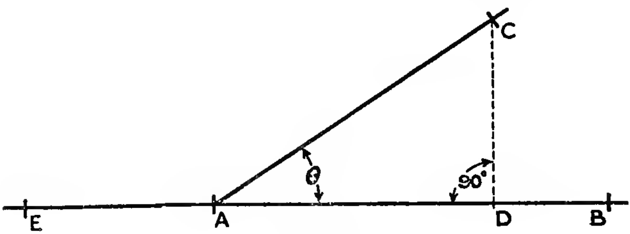

FIG. 66.

of regarding a rectilineal angle as " the inclination of one straight line falling upon another in the same plane," or as " the amount of turning through which one of the lines bounding an angle must pass to coincide with the other bounding line "; thus confining the view to mere lines and diagrams instead of focussing attention upon the actual realities and needs of human life.

Suppose a man were standing at A in Fig. 66 (which represents a plan), with his face turned towards B, having

arrived at A in the course of a journey from, say, E in a straight line towards B. Suppose also that he wishes to proceed from A towards the position C, there being no visible object in or about that position upon which he may fix his gaze to direct his steps. Clearly, it is necessary that he should know the amount by which he must turn his body about its vertical axis at A, from his basis line of vision, AB; and his need is to know *how much turning*, just as in other things his need is to know how much distance, how much force, or how much time.

We need not discuss here the processes by which a conventional "unit of turning" was obtained, nor the relations which that unit (the sexagesimal degree) bears to other measurements in more or less common use. These matters are well and fully described in many excellent books; and if the student will read such descriptions in the light of physical reality—regarding the operations involved always as mere human countings of things in groups, instead of as supernatural processes which defy human comprehension, and apply to the things of this earth in some mysterious way which is at best only an imperfect reflection of the complete ideal—he will find them at once simple, interesting, and helpful. All the records we have show plainly that these things were deduced and devised by ordinary men, working upon the simple and natural process of utilising the results of their common experience and observations to meet their needs, satisfy their aspirations, and assist their researches in other directions. One typical instance is supplied in the "measurement" of the height of one of the pyramids by means of its shadow on the ground, in conjunction with the shadow cast by a vertical stick, at the instant when the stick and its shadow were exactly equal in length; and there are others equally indicative.

Two facts, both highly important, should be observed in passing:

(1) A *vertical line* is a line radiating from the centre of

the earth—from which it follows that no two vertical lines can be parallel (though their divergence cannot be detected by ordinary methods of measurement over practical ranges); and that all vertical lines, except those at and near the poles, swing through considerable angles (with regard to the straight line which may be supposed to pass through the centres of the earth and the sun) in the course of each day; and

(2) A *horizontal line* is a line perpendicular with some particular vertical line—so that horizontal lines are subject to the same conditions and variations as those set out above with regard to vertical lines.

Returning now to the traveller of Fig. 66, it will be clear that, even when the necessary amount of turning is known and effected, only the *direction* AC is obtained. To arrive at the position C, its distance from A must be known; and thus the right-angled triangle ACD, the counterpart of which forms the basis of all trigonometrical calculation, is introduced.

It may be objected that, in many cases, we seek only to know the distance between two points (*i.e.* the side of some triangle in process of solution), without the desire, need, or—as in astronomical calculations—possibility of actual passage from one to the other; but while this is true, it is also true that our measurements (or estimates) of such distances are effected on the same basis, and are expressed in terms of the same units, as though such passage were either necessary or contemplated. And in whatever units an angle may be measured, or by whatever symbols it be expressed, the object is always to either determine or specify the amount of turning required by a man who, proceeding from, for example, the position E in Fig. 66 directly towards B, arrives at the position A, and would proceed thence towards C.

Circular Measure of Angles.—The circular measure of

angles is convenient for some purposes. Though the impracticability of laying off and measuring accurately distances along the circumferences or arcs of circles prevents it from being used in the work of actually measuring angles or setting them out, the student should acquire an intimate acquaintance with angles expressed in radians as well as in degrees ; and should fully appreciate the fact that, whether a man stands at A (Fig. 66) and turns through the angle BAC, or describes a circular arc with A as centre and lays off along this arc a length bearing the appropriate ratio to the radius with which the arc was described, the object and effect are the same, while both operations are mere countings of real things in groups.

As in all ratios, the consequent may be regarded as *one unit thing*, even though it consist of either a fraction of some other thing commonly regarded as itself a unit thing, or an assemblage of such other things. For example, the ratio 3 : 2 becomes the ratio 1·5 : 1 if the two things forming the consequent group of the given ratio be regarded as together forming a new unit thing. Hence, with whatever radius the arc be drawn for a particular angle, if that radius be regarded as a unit distance, and the length of the arc subtending the angle measured in terms of that unit distance, the number of such units (or the fraction of such a unit) in the length of the arc will be equal to the number of radians (or the fraction of a radian) comprised in the angle ; but it is necessary that care be taken to avoid confusion between the *angle* and the *length of the arc*. Much trouble and difficulty have been introduced—in other matters as well as in trigonometry—through failure to keep such essential differences clearly in view. The simplification of ratios by adjusting the magnitude of the unit thing to make the consequent 1 is often very convenient in practice, and it should therefore be used freely —but it must be remembered that *the ratio is still a ratio*, notwithstanding such adjustment of the unit thing.

One radian is simply that amount of turning which is

effected when the length of any arc described about the point of turning as centre, and subtending the turning, is exactly equal to the radius with which the arc was described; and similarly, an angle of, say, 0·2367 radian is that amount of turning effected when, if the arc be drawn with a radius of 10,000 length units, the length of the arc subtending the turning is equal to 2367 of those units.

Properties of Triangles.—Some of the most important properties of triangles become obvious when regarded from the standpoint of physical fact; and the value of the information obtainable by such means is in no way reduced by the simplicity of the processes involved.

For example, suppose a man (represented by an arrowhead with the point forward) walked along the three sides of a triangle as indicated in Fig. 67, starting at A and proceeding towards B. Arriving at B, he will turn through the exterior angle B_1BC, and proceed along the side BC; arriving at C, he will turn through the exterior angle C_1CA, and proceed along the side CA; and finally, arriving at A, he will turn through the exterior angle A_1AB, facing towards B.

Now, in these three turnings he will have faced in every horizontal direction—*i.e.* he will have turned through four right angles—and will stop at A facing, as he started, in the direction of B.

Hence, the exterior angles of any triangle are together equal to four right angles; and similarly it may be shown that the exterior angles of any rectilineal figure (having no re-entrant angles) are together equal to four right angles.

Again, suppose a man had eyes in the back of his head as well as in the usual place, and could walk forward or backward with equal readiness. Let this man walk along the three sides of the triangle indicated in Fig. 68, starting at A and proceeding towards B; but let him turn only through the interior angles of the triangle. For clearness in illustration, the man is represented by a double arrowhead, the backward of which is blacked in.

Arriving at B, he will turn through the angle ABC, and proceed backwards to C; arriving at C, he will turn through the angle BCA, and proceed forwards to A; and arriving at A, he will turn through the angle CAB, and will then be facing directly away from B—*i.e.* he will have turned

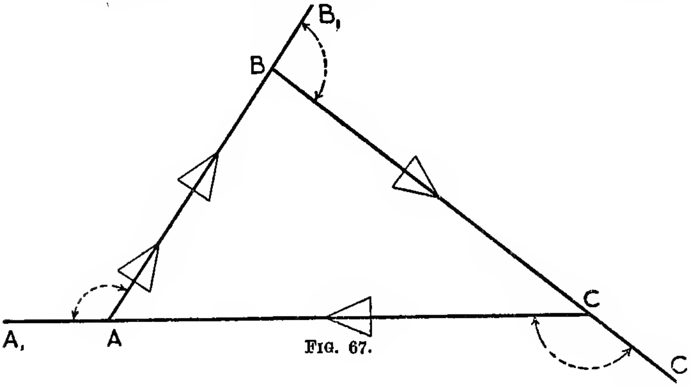

FIG. 67.

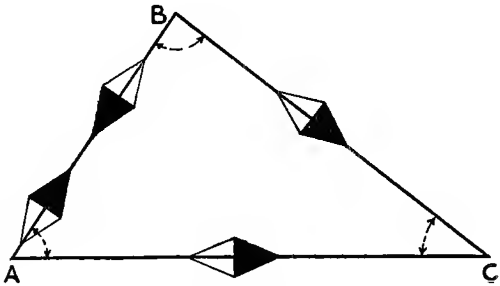

FIG. 68.

through *two* right angles, whence it follows that the three angles of any triangle are together equal to two right angles.

Other important properties of triangles, and of rectilineal figures which may be regarded as assemblages of component triangles, may be observed by similar means. These may be left as exercises for the student, who will assuredly derive real benefit from such investigations if conducted in the right spirit.

Trigonometrical Ratios.—There is no need to discuss here in detail either the evolution of the trigonometrical ratios themselves, or the names which have been given to them—though this is a most fascinating study, and one well worth the while of those who are sufficiently interested; for one may see from it the workings of the minds of men who, though they spoke in a language different from ours as regards externals, and dealt with problems which (on the surface) were not like those of our day, were nevertheless as simple and human as ourselves. The findings of these men have been made to appear mysterious and supernatural; but such records as we have of their actual work show clearly that they were ordinary plain men, working with utilitarian objects, and proceeding always from ascertained fact to logical deduction—and always testing their inferences thoroughly by the appeal to fact before accepting them as sound or putting them forward as reliable.

It must suffice to mention a few points concerning the ratios and their practical utility—points which, for all their simplicity, are too often either missed altogether, or only partially appreciated.

The Tangent of an angle is merely the ratio borne by the "rise" to the horizontal "going" which corresponds with that rise; the Sine is the ratio borne by the rise to the going on the gradient; and the Cosine is the ratio of the horizontal going to the going on the gradient.

The reciprocals of these three ratios—the Cotangent, the Cosecant, and the Secant respectively—are sometimes useful—

(1) In writing, to avoid the use of such symbolism as

$$\left(\frac{1}{\tan\theta}\right), \left(\frac{1}{\sin\theta}\right), \text{and} \left(\frac{1}{\cos\theta}\right); \text{ and}$$

(2) In calculation, to permit multiplication instead of division.

The latter is the sole purpose of their inclusion in tables;

for in calculation by the aid of logarithms, with negative characteristics involved, it is considered by some easier to deal with addition than with subtraction—though it is open to question as to whether the saving effected is worth the extra trouble involved in reference to additional (and inevitably more complicated) tables in ordinary practical work. Even for the calculations of land surveying and setting out works of magnitude—where a greater degree of accuracy is required than in the calculations of ordinary mechanical and structural work—the author prefers to use the three principal ratios only as a general rule; and has found this course at once less troublesome and more expeditious where a large volume of calculation has to be done.

Tables of what are called "Logarithmic" Sines, Cosines and Tangents are available, and are sometimes recommended as a means for saving time in reference, since, it is stated, without such tables, the "natural" sine, etc., must be first obtained from the ordinary tables of trigonometrical ratios, and then its logarithm obtained from the tables of logarithms. After extensive trials of both methods the author is of opinion that, for the great bulk of ordinary work, tables of logarithmic sines, etc., do not provide sufficient advantage to compensate for the additional trouble and care which their use entails. There are always other numbers concerned in the calculations, and it is little, if any, more trouble to take all the logarithms required at once than to separate those which are already logarithmic and seek the logarithms of the remainder; while it is certainly less confusing to work uniformly from *all* numbers to *all* logarithms at one stage, afterwards returning from logarithms to the corresponding numbers at the conclusion.

Moreover, the logarithmic sines, etc., so tabulated are not the true logarithms of sines, etc. The actual ratios being mostly (sines and cosines always, of course) fractions, the characteristics of their logarithms are negative; and

for small angles, the characteristics, besides being large, change rapidly. To avoid the trouble and complication which would be inevitable were these logarithms tabulated with their proper negative characteristics, $+10$ is added to each, making them all positive. Thus, the number tabulated as the logarithmic sine of θ is really the logarithm of $\{(\sin\,\theta)\times 10^{10}\}$.

Hence, before a logarithmic sine, etc., can be used in a computation which is to result in the determination of a distance, 10 must be subtracted to obtain the true logarithm of the sine, etc.; and if the object of the calculation be to determine some ratio of an angle from the logarithms of distances only, 10 must be added to the resulting logarithm before reference can be made to the tables of logarithmic sines, etc.

It will be clear that considerable care is necessary in these manipulations; and since the tables necessary for reference are both more numerous and more complicated than those which would suffice if the ordinary straightforward methods were followed, the latter are preferable —especially as an engineer has to make all sorts of calculations dealing with angles, and a single method which will suit all is more convenient than several methods which have each their own particular range of suitability.

The logarithmic sines, etc., as tabulated (*i.e.* with $+10$ added to the proper logarithms) are denoted as "L Sin θ," "L Cos θ," etc., in contradistinction from "log Sin θ," "log Cos θ," etc.; but even among land surveyors (who would have better reason for using them than would a mechanical or structural engineer) the author has found the logarithmic ratios comparatively little—and much less than formerly—used.

Positive and Negative Ratios of Angles.—The generally taught method of tracing the changes in the ratios of an angle as the angle increases from 0 through the four quadrants to 360 degrees is a source of difficulty to many students. The variations and limitations as regards the

magnitudes of the ratios are usually seen without much trouble; but the statements to the effect that the Sine is positive in the first and second quadrants, and negative in the third and fourth; that the Cosine is positive in the first and fourth quadrants, and negative in the second and third; and that the Tangent changes instantly from positive infinity to negative infinity at 90 degrees, and again at 270 degrees, too often succeed only in conveying the false impression that these ratios are governed by some mysterious and incomprehensible system of "laws."

The fact is that these "changes in sense" are purely conventional; and were introduced solely for the admirable purposes of simplifying the formulae used in the solution of triangles, and rendering the tables of trigonometrical ratios corresponding with angles between 0 and 45 degrees sufficient for all angles throughout the four quadrants.

In reading angles with a theodolite or other similar instrument in the field or on the sea, it is convenient to fix the zero mark to agree with the base line; and to read all angles from it, thus avoiding the need for additions and subtractions of awkward angles under circumstances which are often eminently unsuitable for such processes, and are therefore opposed to the minute accuracy which is essential to such work.

Hence it is convenient to deal with angles ranging through all the four quadrants. To tabulate the ratios of all these angles would, however, be to merely repeat for the second, third and fourth quadrants those already tabulated for the first quadrant; which would render the tables at once more costly and less convenient for practical purposes. Moreover, since the Sine of an angle is the Cosine of its complement, while the Tangent of an angle is the Cotangent of its complement, if the ratios for angles between 0 and 45 degrees be tabulated nothing further can be required as regards magnitudes.

Now, for triangles containing an obtuse angle, the square on the side subtending the obtuse angle is *greater*

than the sum of the squares on the other two sides; while, if the obtuse angle were replaced by its supplement (the lengths of the sides about this angle remaining unchanged), the square on the side subtending the altered angle would be *less* than the sum of the squares on the other two sides. Moreover, the *deficiency* in the second case would be exactly equal to the *excess* in the first case.

Euclid employed two distinct propositions—viz. II. 12 and II. 13—to demonstrate these two cases; while he

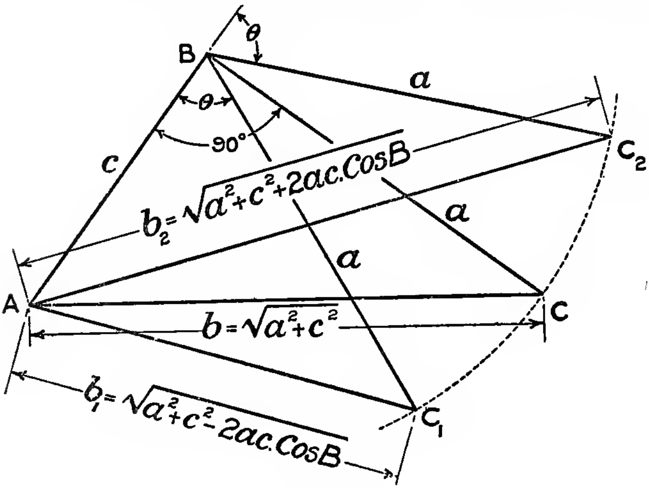

FIG. 69.

employed yet a third proposition—viz. I. 47—to demonstrate the intermediate case in which the angle bounded by the two known sides is neither obtuse nor acute, but a right angle.

The three cases are illustrated in a single diagram in Fig. 69, which should be clear without further description.

In practical work it is more convenient to use the single relation

$$b^2=a^2+c^2-2ac\cos B$$

to cover all three cases, cos B being taken always as the

cosine of the *acute* angle at B—*i.e.* the exterior angle (or supplement) if the actual angle ABC be obtuse.

It would, of course, be possible to use the double sign "$\mp$," rendering the relation

$$b^2 = a^2 + c^2 \mp 2ac \cos B;$$

the negative sign being used when the angle ABC is acute, and the positive sign when the angle ABC is obtuse—and, as a fact, this method is preferred (and used) by the author as being at once more simple and more clearly representing the realities involved. It is believed that the introduction of positive and negative signs for the ratios of angles was due solely to a desire to make the single relation

$$b^2 = a^2 + c^2 - 2ac \cos B$$

serve for all cases; and that the application of signs to the Sine and Tangent followed as a mere logical consequence of the conventional arrangement which was found to serve for the Cosine in the above relation.

The student should convince himself thoroughly of the fact that there is nothing in the ratios themselves in any way concerned with positive (*i.e.* asset-increasing) nature or negative (*i.e.* asset-reducing) nature, except in so far as they affect the lengths of the lines forming the sides of triangles; and that the use of such signs is merely a convention for the purposes of practical convenience—while it is open to question as to whether it has served that object very fully.

The Solution of Triangles.—In any right-angled triangle, such as ABC in Fig. 70, it is obvious that, if the side c and the angle A be known (in which case, of course, all the angles are known, since $B = 90° - A$), the side a is equal to the side c varied in the ratio borne by any "perpendicular" to its corresponding "hypotenuse" for the angle A. To this ratio is given, for convenience in writing, the name "Sin A"; and hence, the relation of the sides a and c may be written

$$a = c \sin A.$$

To simplify the arithmetical process of "varying c in the ratio known as Sin A," we tabulate Sines (and other trigonometrical ratios) as decimal fractions; but they are none the less ratios borne by the numbers of length-units in perpendiculars to the numbers of similar length-units in the corresponding hypotenuses. Thus, when we say that Sin $29° - 20' = 0{\cdot}4899$, we mean only that the angle measured as $29° - 20'$ with a protractor might be equally well specified as an angle which gives 4899 length-units of perpendicular "rise" for 10000 similar length-units of "going" on the gradient. Similarly, when we say that Tan $59° - 40' = 1{\cdot}709$, we mean that the angle might be

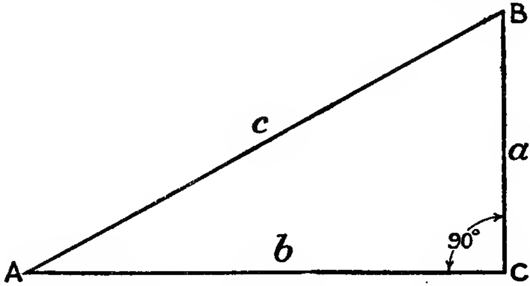

FIG. 70.

specified as that which gives 1709 length-units of perpendicular "rise" for 1000 similar length-units of horizontal "going."

In any triangle, such as ABC in Fig. 71, if the perpendicular BD be drawn from B to intersect the side b in D, the triangle will have been divided into two right-angled triangles, ABD and DBC. Hence it is evident from the foregoing considerations that, with regard to the angle A,

$$BD = c \sin A\,;$$

while, with regard to the angle C,

$$BD = a \sin C.$$

Equating these two values of BD, it follows that

$$a \sin C = c \sin A.$$

Or, having observed that BD $=c \sin$ A, we might further observe that $a:\mathrm{BD}::1:\sin \mathrm{C}$; whence $a=\mathrm{BD}\left(\frac{1}{\sin \mathrm{C}}\right)$. Substituting in place of BD its equivalent ($c \sin$ A),

$$a=c \sin \mathrm{A}\left(\frac{1}{\sin \mathrm{C}}\right);$$

which means that the side a is equal to the side c, varied first in that ratio which we call the Sine of A, and the

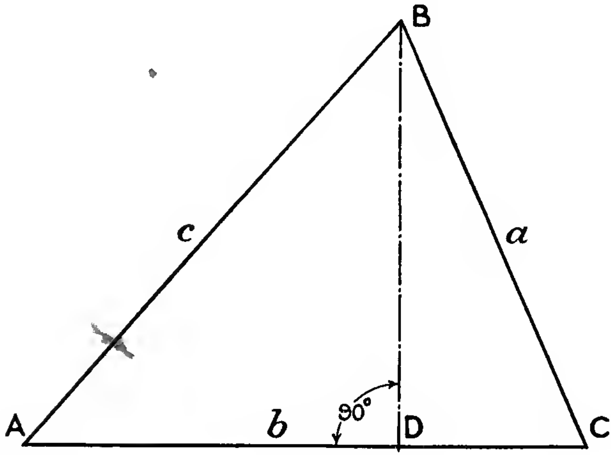

FIG. 71.

resulting length varied in a ratio which is the reciprocal (or inverse) of that which we call the Sine of C.

All the other relations which are useful in solving triangles may be deduced and followed by similar and equally simple reasoning; and if they be thoroughly dissected and analysed on the basis of plain reality, they will become so obvious that there will be no difficulty whatever in remembering them.

It is probable that, in actual work, the student will find that the most generally useful of such relations are:

(1) The Sine Rule—*i.e.* $a \sin \mathrm{C}=c \sin \mathrm{A}$, etc.;

(2) The Cosine Rule—*i.e.* $b^2 = a^2 + c^2 \mp 2ac \cos B$, etc.; and

(3) The Tangent Rule—*i.e.* $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c} \cdot \cot\left(\frac{A}{2}\right)$, etc.

The Tangent Rule is, perhaps, more convenient when expressed in the form:

$$\tan\left(\frac{B-C}{2}\right) = \frac{(b-c)}{(b+c)\tan\left(\frac{A}{2}\right)}.$$

All the other relations, set out and demonstrated in good text-books, should be studied and realised, however, since they give a more complete understanding of the facts than can be obtained if only two or three be studied and used. In particular, the relation for the area of a triangle in terms of its sides alone,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

where Δ is the area of the triangle, a, b and c the lengths of its sides, and s the "semi-sum of the sides"—*i.e.* $s = \frac{a+b+c}{2}$, is of frequent use in surveying with a steel tape or chain only, and where great accuracy is necessary in estimating the area.

For example, building sites in large and important towns are often very irregular in shape, and so small and congested that the observation of angles with instruments is impracticable and useless; while land values are so high that every square foot of area represents a considerable sum of money. The plan of such a site is indicated in Fig. 72; and it will be seen that if all the seven bounding lines be measured accurately, together with the dimensions indicated by the four dotted lines, the site will have been divided into five triangles, all three sides of each being known. The area so estimated will obviously be far more reliable than one obtained by scaling from a drawing, no matter how carefully the drawing may be made.

The Ambiguous Case.—So much is made of the Ambiguous Case in text-books that the thing is apt to be regarded as a sort of evil spirit, which may appear at any moment to trap the unwary.

The ambiguous case arises only when the information available is insufficient for the complete determination

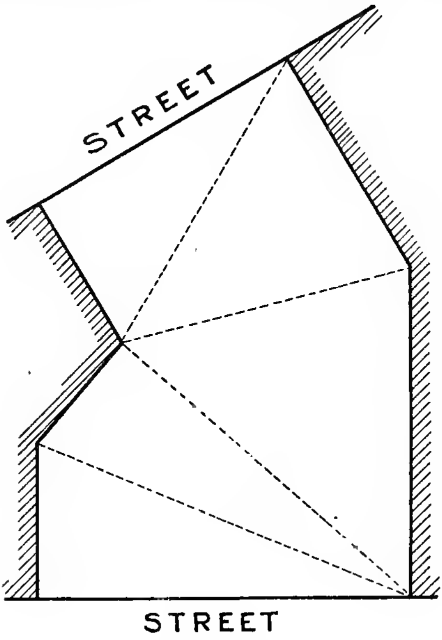

FIG. 72.

or specification of a triangle; and this means that the information has not been taken properly.

In surveying—and no less in setting out works of magnitude—the best way to deal with the ambiguous case is to take such precautions as will ensure that it shall not arise; and this may be effected by refusing always to be satisfied with information which covers less than two angles of a triangle, unless the single angle observed

be the angle bounded by two known or measured sides of the triangle; or unless all three sides are known. Even in these cases, moreover, it is well to have information concerning an additional angle if possible (and with instruments available, and conditions suitable for the measurement of angles, it is seldom that a case arises where this is not possible, granted the desire and willingness to take a little trouble), so that an independent check may be applied at the time when it is both easiest and most effective.

Ratios of Two Angles.—The ratios corresponding with angles which are themselves the sum or difference of two other angles, expressed in terms of the ratios for the two component angles, are frequently of use in practice; and the student will do well to acquire a clear understanding of their reality and significance.

Of the six distinct relations (which may be summarised into three expressions by the use of the double sign),

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B;$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B; \text{ and}$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B},$$

it will suffice if we investigate one—viz. that for $\sin(A+B)$—indicating the line of argument, and leaving the others as exercises for the student.

From Fig. 73 it will be clear that, in addition to the main triangle OPR, there are three distinct right-angled triangles—viz. OST, OPS and VPS—in which the known angles A and B occur. The triangles OST and VPS are similar, both containing the angle A. Ratios may, therefore, be arranged in any of these three triangles.

Obviously:

$$\sin(A+B) = \frac{PR}{OP} = \frac{PV+ST}{OP} = \frac{PV}{OP} + \frac{ST}{OP}.$$

Now, PV belongs to the triangle VPS, and OP to the triangle OPS. Hence, this ratio is not convenient as it stands, inasmuch as it does not relate to either of the

component angles. If a line could be found belonging to *both* of these triangles VPS and OPS, the ratio PV : OP might be compounded by the insertion of the desired line as an intermediary, this line forming the consequent of a ratio with PV as antecedent, and the antecedent of a

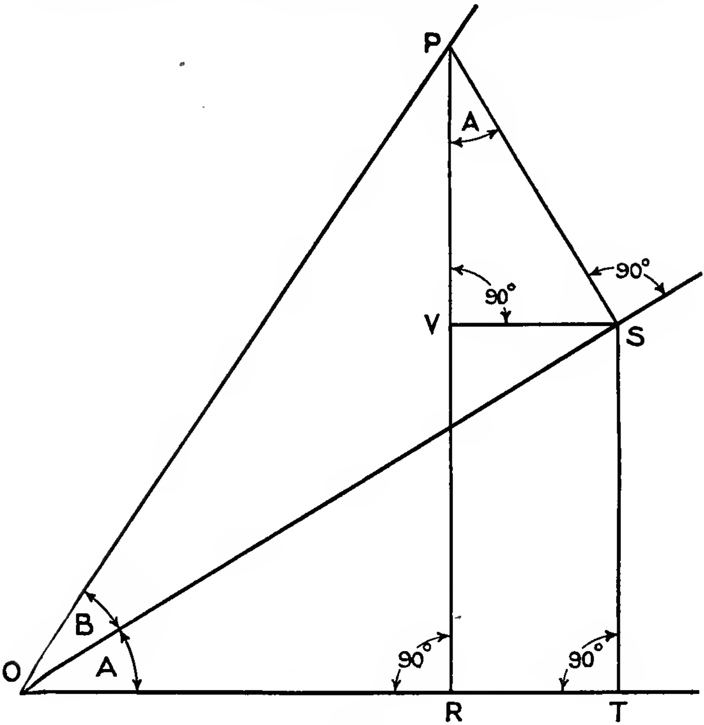

FIG. 73.

second ratio with OP as consequent; these two ratios necessarily relating to the component angles in some way.

The expression might then be written :

$$\sin(A+B) = \frac{PV}{} \cdot \frac{}{OP} + \frac{ST}{OP},$$

in anticipation of the desired line being found.

Similarly ST belongs to the triangle OST, and OP to the triangle OPS. Hence, this ratio is also of little service as it stands; but if a line could be found belonging to both of these triangles OST and OPS, such line might be used to form the consequent of a ratio with ST as the antecedent, and the antecedent of a second ratio with OP as the consequent. In anticipation of such a line being found, the expression might be written:

$$\sin (A+B)=\frac{PV}{\quad}\cdot\frac{\quad}{OP}+\frac{ST}{\quad}\cdot\frac{\quad}{OP}.$$

There is but one line—viz. PS—which belongs to both of the triangles VPS and OPS; and only one—viz. OS—which belongs to both of the triangles OST and OPS. Inserting these appropriately, the expression becomes:

$$\begin{aligned}\sin (A+B)&=\frac{PV}{PS}\cdot\frac{PS}{OP}+\frac{ST}{OS}\cdot\frac{OS}{OP}\\&=\cos A \sin B+\sin A \cos B.\end{aligned}$$

On examination of this reasoning and inference, it will be seen that Sin (A+B) is but the sum of:

(1) Sin A, varied in that ratio which we call Cos B; and

(2) Sin B, varied in that ratio which we call Cos A; a fact which is really quite obvious from Fig. 73 if regarded in the right way.

Let the length of OP be considered as 1 unit. Then the fraction of such a unit in the length PR will be equal to the Sine of (A+B); while the fraction of such a unit in the length PS will be equal to the Sine of B, and the fraction of such a unit in the length OS will be equal to the Cosine of B.

Now, PR = PV + ST;
But PV = PS cos A = sin B cos A; and
ST = OS sin A = sin A cos B; whence, as before,
$$\sin (A+B)=\sin A \cos B+\sin B \cos A.$$

The latter method of reasoning is obviously both more

direct and more strictly consistent with the information given than the former method.

Similar reasoning may be readily applied to the cases for Sin $(A-B)$, Cos $(A+B)$ and Cos $(A-B)$.

Attention should also be given, in the same way, to the relations corresponding with the sum and difference of two Sines, and the sum and difference of two Cosines, these being frequently of use in practical work, both of themselves directly, and as links in the chains of argument relating to other matters.

CHAPTER XI

DIFFERENTIATION AND INTEGRATION

The Calculus in General.—So many really intelligent students are mystified by the "Calculus" that the author feels called upon, even at the risk of being thought tedious or parrot-like, to repeat here what has been stated many times already in the preceding pages—that the only process involved is that of simply counting real things in groups. Also, to anticipate the inferences of this chapter by stating that a "differential coefficient" (*i.e.* a comparison of differences) is but a *rate of variation*—a change of so many things in one group for a change of one thing in the companion group; and that an "integral" is merely a counting of things accruing to or withdrawn from one group by reason of a specified change in the numerical contents of a companion group, the rate of variation being known.

These processes are just as real, physical and simple as are all those which we have discussed. They involve no principle other than that of addition; and they may be performed actually, with real things employed as symbols. Moreover, for the purposes of practical engineering in general, only a small range of the calculus is required; little more being necessary beyond the differentiation and integration of x^n and the simple trigonometrical ratios. To be of true service, however, the little that is required must be known and understood as an actual and living reality, with its scope and limitations clearly comprehended, and its application to appropriate cases available with

the utmost readiness, facility and certainty; but there is no need for difficulty in the acquisition of such knowledge and control, for (as will be seen presently) all the processes involved may be watched in actual operation at the cost of a few pence and a little trouble, leaving intelligent practice in their application as the only other requisite for facility.

Much of the difficulty experienced by students in acquiring a working command of the calculus is due to a lack of clearness regarding its purpose and utility in practical calculations; and it is unfortunate that so many enthusiasts demonstrate the "convenience" of the calculus by applying it, at considerable length and in the most elaborate detail, to cases in which its use is totally unnecessary—if not irrelevant. The author desires to be quite frank with the student, on this as on all other points; but it must be clearly understood that statements so made are no more than expressions of opinion based upon a fairly wide experience in practical engineering work, alike in design, fabrication and administration. They are not intended (and must not be taken) as dogmatic statements covering the whole range of work to which mathematics may be applied. The needs of an individual man in his daily work can only be appreciated properly by the man himself; and the acquisition of powers to meet those needs is therefore, of necessity, a task for the man himself also—indeed, it is but the development of his own personality, in which the interference of others is at once impossible and intolerable. With this proviso, it may be stated that the main purposes served by differential coefficients are:

(1) The location of turning points (usually maxima or minima) in the numerical contents of groups which vary with other groups; and
(2) The determination of instantaneous rates of variation which are common to two sets of varying groups.

An instance of the former type is supplied in the determination of the most economical proportions for tanks and containers of various shapes,[1] and in the location of sections in beams at which maximum deflection occurs;[2] while an instance of the latter type occurs in the location of points of contraflexure in propped cantilevers and continuous beams.[2]

It is both interesting and instructive to note that in the bending of initially straight beams, four distinct groups of things are related, all varying dependently and with regard to the distance between the particular section under consideration and that over a support. These four groups comprise: (*a*) the shearing forces; (*b*) the bending moments; (*c*) the slopes of the axis as bent; and (*d*) the deflection (or displacements perpendicular to the line of the beam axis before straining) of the beam. The rates of variation for these four groups with regard to the distances along the beam are useful in designing, by reason of the facts that:

(1) Where the shearing force is zero (*i.e.* a minimum), the bending moment is a maximum;
(2) Where the bending moment is zero (*i.e.* a minimum), the slope is a maximum; and
(3) Where the slope is zero (*i.e.* a minimum), the deflection is a maximum.

Also, at a point of contraflexure, the slopes of the portions to right and left of that point are equal; and such points may often be located more easily from this fact than in any other way. Their location is desirable for a variety of reasons in practice—notably that joints in the flange members may be disposed where they will cause the least possible waste of material and labour.

[1] See the author's book on *Tank Construction* (Emmott & Co. Ltd.).

[2] See the author's book on *Structural Steelwork* (Longmans, Green & Co.).

Similarly, integration is useful in many cases of mensuration which could not be satisfactorily determined by any other means; and it is useful in other practical work —notably in the bending of beams, where the slope of the axis at a particular section is estimated by adding all the increments of bending moment which have accrued to that section, and the deflection at a particular section is estimated by adding all the increments of slope.

These are but a few instances of the purpose and utility of the calculus in the calculations of practical engineering. The others are, however, all as plain, straightforward and real as those cited above; and the reality which we shall presently demonstrate belongs equally to all differentiation and integration concerned with actual things.

Experimental Differentiation and Integration.—By means of the simple apparatus here described, the differentiation and integration of groups belonging to the x^n type, with regard to x, may be observed in actual operation over a range sufficiently wide to cover most of the cases likely to arise in practical work, and sufficiently wide also to establish the general relations applicable to all cases of such grouping.

Take two thin sheets of wood, each about 10 in. in height and 5 in. in width. Cut in one a circular aperture, 1 in. in diameter, as shown at (*a*) in Fig. 74; and in the other, a vertical slit, 1 in. in height and $\frac{1}{2}$ in. in width, as at (*b*) in the same sketch. These two apertures should be on the vertical centre-lines of the sheets, and their centres at the same height (about 7 in., as shown) above the bottom edges.

Then take two straight laths, each about 5 ft. in length, and 2 in. by 1 in. in section. Frame these, with end-pieces, cross-ties and diagonal braces of the same material (or, of course, by any other convenient means to give the same effect), to form a rectangle about 1 ft. in width, as shown in Fig. 75. Mark both of these laths, as shown, in unit divisions of 10 in., each division subdivided into tenths—

and, if desired, into hundredths—and lay them upon a table-top, as indicated in Fig. 76.

Mount the two wooden sheets (or "screens," as we may call them) of Fig. 74 upon suitable bases. Fix that having the circular aperture so that it shall be truly over the zero marks on the two scale-laths; and fix that having the rectangular slit over the marks 1, as indicated by A and B respectively in Fig. 76.

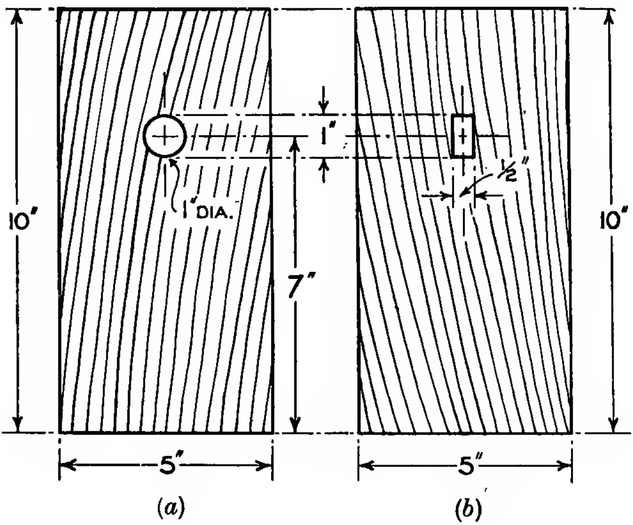

FIG. 74.

Take a small and light drawing-board, on which sheets of paper may be pinned—or a frame in which may be placed sheets of Bristol-board—to serve as "slides"; and mount this on a base possessing sufficient stability to hold the board or frame truly vertical, with its plane perpendicular to the scale-laths, and having wheels or other fittings to permit the board to be moved easily backwards and forwards along the table-top, in a direction parallel with the scale-laths. Place this board or frame astride the

scale-laths in any position such as C in Fig. 76, with a suitable "slide" mounted in it. The base or carriage may be cut away or otherwise adapted to receive two thin metal strips which will truly indicate, one on each of the scales, the distance of the "slide" from the zero mark, as shown.

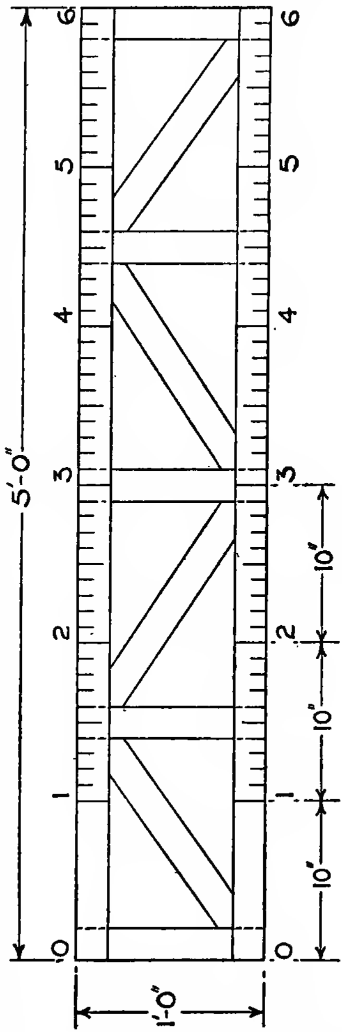

Fig. 75.

If one eye be applied to the circular aperture in the screen A, a portion of the "slide" at C will be visible through the slit in the screen B; the visible area being sharply defined by the edges of the slit, which may be made to yield better definition by bevelling them.

The screen A serves to fix the position of the eye with proper relation to the screen B, and should be so used. If light be excluded from the space behind B, and the "slide" brightly illuminated, the observations will be more easily made, and also more accurate. This may be effected by means of a black cloth (such as is used by photographers in focussing) fastened to the top and sides of B, and falling

well over the head; but if this be used, it will probably

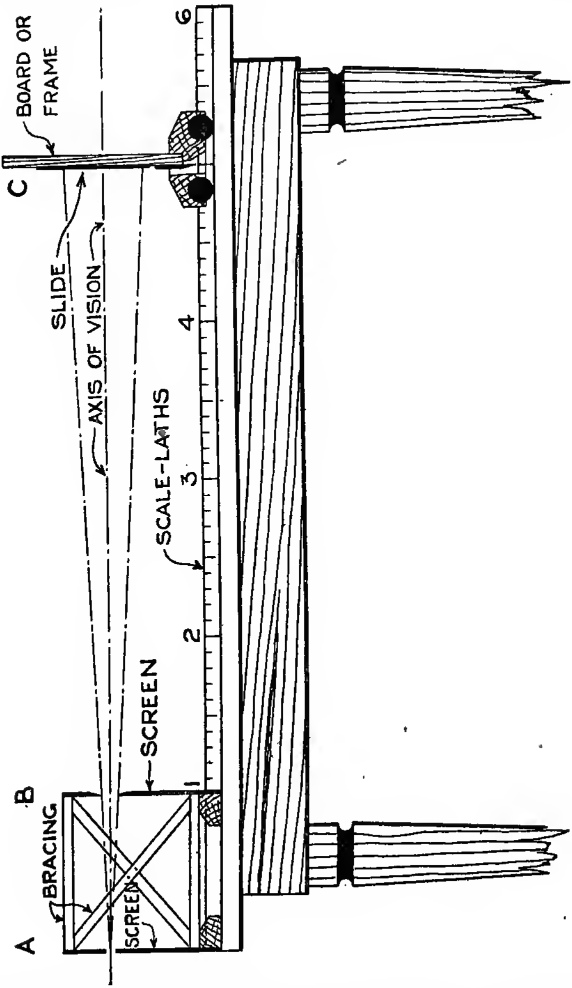

FIG. 76.

be necessary to provide bracing, stays or other lateral support for the screens A and B, as indicated in Fig. 76.

All the constructional details for the apparatus are left for the student to devise; and the conditions to be provided for will be seen clearly after the following description of its use has been read. Obviously, the more care taken to ensure clear definition and accuracy, the more satisfactory will be the results obtained. There is no need for elaboration or high-grade finish, though some students have thought sufficiently well of the idea to spend much time and trouble in the production of extremely neat and durable examples of the apparatus, while others have obtained a firm and serviceable grasp of the calculus and its ways by making the apparatus in imagination only.

The screens A and B should be carefully adjusted so that both are truly vertical, their planes perpendicular to the scale-laths, and their apertures co-axial on a horizontal line parallel with the scale-laths. Also, the board or frame C should be provided with means for adjusting the height of the "slide" for convenience in observation.

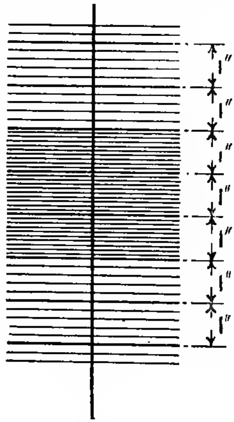

Fig. 77.

A few preliminary observations and inferences may be made concerning the apparatus and its mode of employment, and then we shall be ready to commence the work of differentiation and integration with it.

Let the first "slide" consist of a series of firm horizontal lines, 1 in. apart, as indicated in Fig. 77. Fine lines 0·1 in. apart should be drawn between the firm lines, as shown; and a single firm vertical line at the middle of the sheet. Mount this on the board or frame C, and adjust the height so that, with the "slide" in contact with the screen B, the upper and lower edges of the slit in the latter coincide with two of the firm horizontal lines of the slide.

As the "slide" C is moved away from the screen B, farther from the eye at A, more paper will become visible through the slit; and a few observations will suffice to show that the case is one of similar triangles, as indicated in Fig. 78, so that

$$H : h :: D : l;$$

whence

$$H = h\left(\frac{D}{l}\right).$$

Now, since the scale-laths are marked in unit distances equal to l, the reading of the scale at C will indicate the number of such units in D; and hence, H will be equal

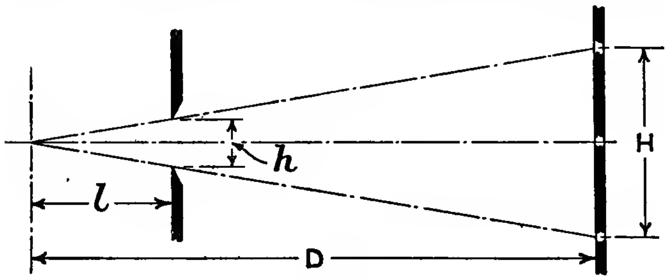

FIG. 78.

to the height (h) of the slit in B, repeated to as many times as l is repeated in D.

Thus, we have two groups of things related: (1) the number of unit distances (each equal to l) between the eye at A and the "slide" at C; and (2) the number of strips on the slide visible through the slit in the screen B—and any alteration in the first of these produces a corresponding change in the second. The two variations are of the same sense—*i.e.* an increase in D produces an increase in H; and this is conveniently termed "direct" variation, in contradistinction from "inverse" variation, in which an increase in one group produces or accompanies a decrease in the other.

Let us see what is the "rate of change" between these

two variable groups—*i.e.* let us find the rate at which H varies with respect to D.

There is no need to describe the operations in detail. It will be found, of course, that the rate of change is *one strip per unit change in* D; and that this rate is constant. If the slide be placed at the mark 2, two strips will be visible, and if then the slide be moved to the mark 2·1, the two visible strips will be increased by two narrow strips, each 0·05 in. in width, one at the top and one at the bottom, giving an increase of 0·1 strip for an increase of 0·1 in D —which is the same rate as before.

The operation of determining this rate of change is called "differentiation," for the simple reason that it is a *comparison of differences*. The rate, "one strip of slide per unit change in the distance D," is called the *differential coefficient of the visible strips with regard to the distance* D, and might be denoted by the symbol $\frac{dH}{dD}$.

This is the whole principle of the Differential Calculus; and, as will be seen, the principle is exceedingly simple.

One fruitful source of difficulty should be noticed. It is the common practice of writing and evaluating differential coefficients as though they were mere abstract numbers. Such a case as that considered above would be regarded as the differentiation of $y=x$; and the differential coefficient would generally be written

$$\frac{dy}{dx}=1.$$

This may be suitable for memorising (though memorising will be unnecessary when a clear view of the realities has been obtained); but whenever it is used for practical calculation, the "things" to which it relates should be specified. It may not always be vitally necessary (though it is frequently worth while) to write them down; but in a case such as that described above, for instance, the expression $\frac{dy}{dx}=1$ should be read—at least in thought—

as meaning *the visible strips vary at the rate of* 1 *strip per unit change in the distance* D; and similarly in all other cases.

It will doubtless have been observed that the differentiation performed with the "slide" of Fig. 77 is that of the linear relation; and, as has already been pointed out, the calculus methods are not necessary for such cases. For the sake of consistency, however, and to show the working of the apparatus suggested, the relation is worth observation a little more fully.

Suppose we were told that, with an apparatus like that of Fig. 76, it had been found that the visible strips varied at the rate of 1 strip per unit change in the distance D; and were asked to state the number of strips which would become visible while D increased from 2 to 5 units—*i.e.* while the slide was moved from the marks 2 to the marks 5 on the scales.

We could find out by observation with the apparatus, of course; but we might easily arrive at the proper answer by means of simple argument, thus: There are to be $5-2=3$ unit changes in the distance D, and each will cause 1 strip to become visible. Hence, when the slide is at the 5-mark, there will be 3 more strips visible than when it was at the 2-mark.

This is called "Integration"; and the Integral Calculus involves nothing more than this principle.

The integral for this case of $\frac{dy}{dx}=1$ is written

$$y=\int_{x=2}^{x=5} 1.\,dx\,;$$

which means neither more nor less than that, knowing the contents of the y-group to vary at the rate of 1 thing per unit change in the x-group, we seek to find how many things the y-group comprised when x was 5, and how many when x was 2, the total variation in the

y-group being, obviously, the difference between these two contents.

In considering the question as to how many things the y-group will contain when x is 5, it must be remembered that we do not yet know how many strips would be visible if x were 0; and that in some cases there may be a number of things so placed that they are not affected by any change in x—as, for instance, if a few strips were painted on the screen B, below the slit, on the side nearest to the eye. To provide for this possible number of things when we desire to find an integral up to a specified value of x, we write the integral as

$$y = \int 1 . \, dx = (x + C) \text{ things},$$

the C denoting a *constant of integration*; and this must always be done, because the ascertained rate of variation (on which the integration is based) can take no account of things in the y-group which are not affected by any change in x.

The "constant" is constant only in that it does not vary during the changes in the particular case under consideration. The symbol C means, of course, a number of *things*, all similar to those which are being added to or withdrawn from the y-group, but so placed that they are not influenced themselves by the changes.

To evaluate C, we notice that it is the value of y when x is 0; and we must be able to ascertain this value from the facts and circumstances of the case under consideration.

In the case of the visible strips with the apparatus of Fig. 76, we cannot bring the "slide" nearer to the eye than the screen B, which stands at the marks 1 on the scales. We can, however, determine the value of C by means of the following argument.

The screen B is 10 in. from the eye; and the slit in it is 1 in. in height. The same extent of "slide" would be visible (with the slide in any particular position) if the

screen B were replaced by another having a slit $\frac{1}{2}$ in. in height, standing at 5 in. from the eye, or by any other screen placed at 10 times the height of its slit from the eye. Hence, if the distance between the screen and the eye were exceedingly small—and we cannot annihilate it, of course—the height of the slit would have to be even more exceedingly small. Carrying the argument to its logical conclusion, it follows that if the distance l (Fig. 78) could be reduced to such an extent that it were smaller than we could perceive, the height of the slit also would be smaller than we could either perceive or conceive; and this we describe by stating that when l is 0 (reading 0 as meaning *smaller than we can perceive*), h is 0. If the slide were brought close up to so minute a slit, the extent of the slide visible would be equal to that of the slit—*i.e.* zero—and if there were no strips painted on the screen B below the slit, we should be justified in stating that y is 0 when x is 0. Hence, the value of C in this case is 0.

Such is by no means always the case, and it will be found that there is often scope for much care and skill in integrating, to so arrange that the value of the "constant of integration" shall be as simple and as easy of determination as possible. The final result is, of course, the same for any case, no matter what course be followed (provided the work be correctly executed); but to drive straight ahead, without thought for convenience or suitability, may involve much work in the simplification of expressions which might be avoided by an intelligent and judicious choice of the course to be followed. One could, of course, travel from London to Cardiff by way of Japan and America; but more expeditious routes are available.

One point, of the utmost importance, should be noticed in passing.

Instead of reasoning as above, it is sometimes argued that if we were to *annihilate* l, h would be actually 0. Now, this supposition cannot be justified, and is therefore not acceptable. If we had the power to (and did) anni-

hilate l, we should at the same time have annihilated the whole apparatus and set of things; and we could therefore no longer make deductions from the facts which obtained before such annihilation took place. If it be contended that the inference is drawn from argument based upon the assumption that l can be annihilated without affecting the groups of things, the contention still falls through bad logic. Knowing l as one of a set of properties relating to two similar triangles, we can determine the consequences of any increase or decrease in l *so long as it exists*. Once permit l to become non-existent, and we have no justification whatever for assuming that it will, when non-existent, exercise an influence similar to that it had when it existed. The importance of clear vision on this point cannot be exaggerated.

To proceed, let each horizontal strip on the slide be divided into some number of equal sub-strips. Clearly, if there be 10 sub-strips to each original strip, as in Fig. 77, these sub-strips will, on movement of the slide, become visible (or invisible) at the rate of 10 sub-strips per unit change in the distance D; and generally, if each original strip were divided into a equal sub-strips, these would vary at the rate of a sub-strips per unit change in the distance D.

Hence, the relation between the groups being

$$y = ax,$$

the rate of variation of y with respect to x is

$$\frac{dy}{dx} = a \text{ things per unit change in } x.$$

Again, having given that the rate at which the y-group varies with respect to x is $\frac{dy}{dx} = a$ things per unit change in x, we could easily integrate, by simple argument from the facts, to find

$$y = \int a \, . \, dx = ax + \mathrm{C};$$

and in circumstances similar to those of the first case C will again be 0, giving the relation as

$$y = ax.$$

If it be desired to observe the differentiation and integration of this relation with the apparatus of Fig. 76 when a is a fraction, it is only necessary to take *larger* strips instead of smaller—to *repeat* the original strips, instead of subdividing them, to form the new strips for observation as "things of the y-group."

We will now proceed to the

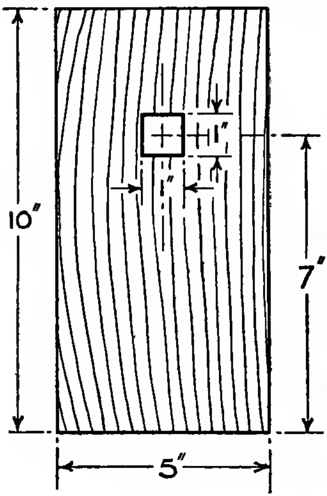

FIG. 79.

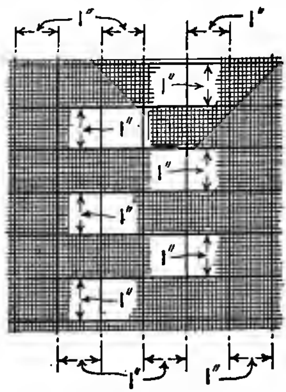

FIG. 80.

differentiation and integration of the quadratic relation—*i.e.* of grouping of the second order, the dependent group comprising as many *rows of things* as there are *things in each row.*

In place of the screen with a narrow vertical slit at B, make another screen having an aperture square in shape, the sides being 1 in. in length, as indicated in Fig. 79.

Replace the former "slide" also by a new one containing a number of squares having 1-in. sides, as shown in Fig. 80. Faint lines may be drawn, 0·1 in. apart, between the firm lines, giving 100 sub-squares to each firm square.

Observation with this slide will show that, as in the previous case, the number of "things" (each thing being represented by a visible square) varies with the number of units in the distance D; but the rate of change is no longer constant. When the slide is at the marks 1 on the scales (*i.e.* close against the screen B), a single square is visible; when the slide is at 2, there are 4 squares visible; when at 3, there are 9 visible squares; and so on. Either by observation or from the properties of similar triangles, it is easily seen that an increase in D produces a twofold increase in the number of visible squares; for it increases the number of *rows*, as well (and at the same rate) as the number of squares in each row. Hence, the number of visible squares varies as the square of the number of units in the distance D; and since the divisions on the scales are equal to those vertically and horizontally on the slide, and the aperture in the screen B is equal to one square on the slide, it is evident that the relation may be written

$$N = D^2;$$

or, for general expression,

$$y = x^2.$$

The student will find most instructive exercises in a consideration of the effects introduced by working to different scales—say, centimetres for the scale-lath divisions, inches for the slide divisions, and 0·7 in. for the sides of the square aperture in the screen B—the variation then becoming of the type $y = ax^2$, which will be considered more fully in due course.

Since the visible squares do not vary at a constant rate for uniform changes in D, the method of taking and interpreting the observations cannot be quite the same as that which served for the linear relation. The slight modification necessary is, however, one of detail only, and not of principle—indeed, it will be apparent that it is merely a broadening of the view, the simpler case being a particular application of the general operation.

With the slide at any position on the scale, the number of visible squares consists of a square-group, there being as many rows, and as many things in each row, as there are units in the distance D. Hence, denoting the number of visible squares as y, and the number of units in the distance D as x, the relation $y=x^2$ holds for all positions of the slide.

Now let the slide be placed at some convenient mark on the scale—say 5. There will be 25 visible squares—5 (because the mark on the scale is 5) rows, with 5 in each row. If the slide be moved to the 5·1 mark, the visible squares will be increased, presenting the appearance of Fig. 81, in which the portions of squares rendered visible through the increase of D from 5 to 5·1 are shown hatched.

To simplify comparison, let the hatched portions be collected about two adjacent sides of the square-group, the appearance then being as indicated in Fig. 82—but it must be remembered always that the hatched area represents a *number of things* added to the assemblage.

The additionally visible squares consist of two rows, each 5 in length and 0·1 in width, and one corner piece which is one-hundredth part of a complete square. This corner piece is somewhat of a disturbing element; for the things in the two main rows about the square-group will vary simply with x and the increment in x, while the corner piece will vary as the *square* of the increment in x.

It should be noted that the information so far obtained is perfectly in order, and will give the rate of change accurately *for the range of the observation.* The rate so indicated will, however, apply only to that particular range; and, further, it will apply only to that range *as a whole.* In consequence of x being increased from 5 to 5·1, the visible squares increased by 1·01; and hence the effective rate of variation over this range *as a whole* is $1{\cdot}01 : 0{\cdot}1 = 10{\cdot}1$ squares per unit change in x.

If the slide be placed at the mark 3, and then moved to 3·1, the visible squares will be varied from 9 to 9·61;

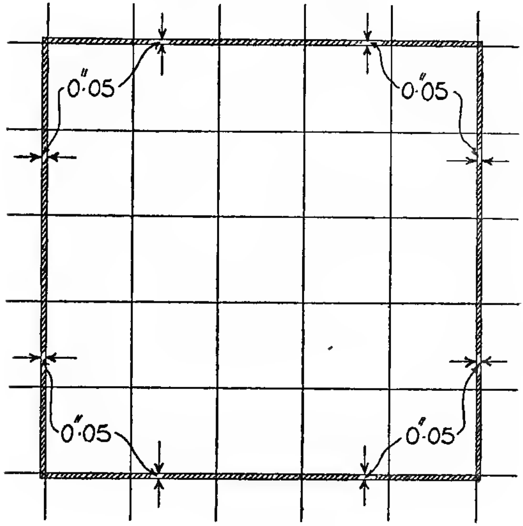

FIG. 81.

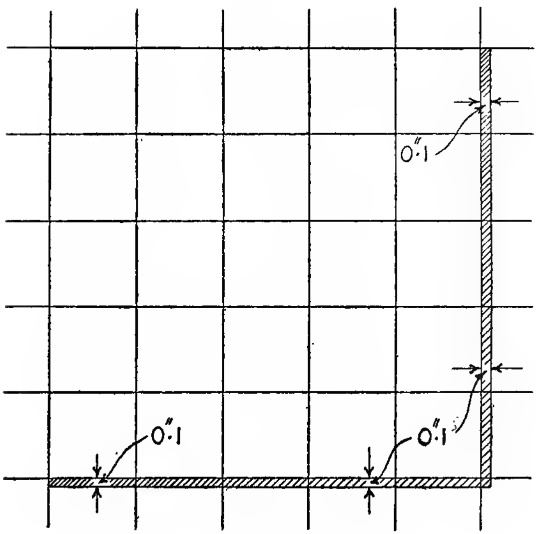

FIG. 82.

the increase being 0·61 square for 0·1 increase in x, giving the rate of change, *for this range as a whole,* 6·1 squares per unit change in x.

Similarly, if the slide were placed at the mark 4, and then moved to 4·1, the rate of variation for this range as a whole would be 8·1 squares per unit change in x.

To obtain a general expression, we might denote the *difference in* x as δx, and the corresponding *difference in* y as δy. Then, for the quadratic variation,

$$\delta y = 2x \,.\, \delta x + (\delta x)^2\,;$$

and the rate of change in y with respect to x will be

$$\delta y : \delta x :: \{2x \,.\, \delta x + (\delta x)^2\} : \delta x,$$

whence

$$\delta y : \delta x :: (2x + \delta x) : 1.$$

That is to say, the rate of change in y with respect to x is $(2x + \delta x)$ things per unit change in x; and this has the obvious inconvenience entailed by its dependence upon the increment in x, rendering it variable, not only for different ranges, but also in the same range if that range be traversed in a single increment or in a number of component increments.

The *instantaneous* rate of variation may be determined by application of the argument used above regarding the effect of an extremely small slit in the first screen B at an exceedingly small distance from the eye.

If some power higher than ours were to make the increment in x smaller than we could perceive, the corner piece in Fig. 82 would be smaller than we could perceive as compared with the two main increment rows about the square-group—though these also would have been reduced to inconceivable narrowness.

Here again it must be noted that, even if we *could* annihilate δx, it would be foolish to do so; for we know nothing as to the influences or effects (if there be any) of a non-existent variation in x. What we may properly

argue is that, if δx were made inconceivably small in comparison with x, then it would be even more inconceivably small in comparison with $2x$; and since δy would have been automatically reduced in the same ratio as had δx, the instantaneous rate of variation might be written as

$$\frac{dy}{dx}=2x.$$

The change of notation from δy to dy, and from δx to dx, is employed to mark the passage of these previously *perceptible* differences into differences of *inconceivable smallness*.

Here it is well to observe a fact which, though of the utmost importance, receives little or no attention. *The differential coefficient is useless for ordinary arithmetical computation of the change in y consequent upon a change in x; for it is true only so long as the alteration in x is inconceivably small, and ceases to be true immediately a perceptible value—no matter how minute—is assigned to the alteration in x.* It has, of course, great utility for other purposes; but it is important that its limitations be known also. To apply the rate of variation, by the ordinary arithmetical process of repetition, to determine the change in y consequent upon a specified change in x, the rate given by $\delta y : \delta x$ must be used. If the instantaneous rate of change or "differential coefficient" ($dy : dx$) be used, it is necessary to employ the method of integration, for reasons which will be clear presently.

Let each inch of the slide be divided into any number (say n) of equal parts, having the effect of dividing each original 1-in. square into n^2 sub-squares. The number (y) of such sub-squares visible for any number (x) of units in the distance D will be

$$y=n^2x^2.$$

Any increase in x will bring more sub-squares into view, and a little consideration will show that

$$\frac{dy}{dx}=2n^2x;$$

whence it is usual to write that, if

$$y = ax^2,$$

then

$$\frac{dy}{dx} = 2ax,$$

the a being, of course, the number of sub-squares in each original 1-in. square, so that each inch of the slide, horizontally and vertically, must be divided into $\sqrt{a}$ equal parts.

Now let us examine the integration for this case. If we are told that one group of things (y) varies with another group of things (x), and that the instantaneous rate of change $\frac{dy}{dx} = 2ax$, we are also told, by inference, that the relation connecting y and x is

$$y = ax^2 + \mathrm{C}.$$

The 2 and the x in $2ax$, by reason of their association with $\frac{dy}{dx}$, tell us plainly that the variation is that of a square-group of things and the things in each row of that group; and the a tells us just as plainly that each thing called x in the square-group has been obtained from some other thing by dividing each row of these latter things, and also each of these things in their rows, into $\sqrt{a}$ equal subdivisions. The mere statement $\frac{dy}{dx} = 2ax$ immediately calls to mind our apparatus of Fig. 76 and the observations given by it with the slide of Fig. 80. The $2ax$ must be the combined length of the two strips lying along the sides of the square-group; and further, their width is δx, while there is also a corner-piece $\delta x \times \delta x$, it having been supposed that δx has been reduced to inconceivable smallness. Although these other factors and terms are not mentioned in the expression, they are none the less plainly and imperatively implied.

Hence we know that if $\frac{dy}{dx}=2ax$, then

$$y=\int 2ax\,.\,dx=ax^2+\text{C},$$

and
$$y=\int_{x=3}^{x=7} 2ax\,.\,\delta x=\{a(7)^2+\text{C}\}-\{a(3)^2+\text{C}\}$$
$$=a(7^2-3^2)=40a.$$

The differentiation of the relation

$$y=ax^3$$

may be observed easily with the apparatus of Fig. 76 if the " slide " be made to show cubes of 1-in. edges (each cube representing some " thing ") in rows and layers of rows, in isometric projection, as shown in Fig. 83 ; and the aperture in the screen B made to the outline of one of the cubes—*i.e.* a regular hexagon, with 1-in. sides—as shown in Fig. 84.

Now, with the slide touching the screen B, 1 cube will be visible ; with the slide at the marks 2 on the scale-laths, 8 cubes will be visible ; at 3 on the scale, 27 cubes ; and so on. The number (y) of visible cubes will be equal to x^3 because an increase in x produces an equal increase in the number of *layers*, in the number of *rows* in each layer, and in the number of *things* in each row.

If the cubes represented on the slide were each sub-divided into a number of smaller cubes, or if the aperture in the screen B were enlarged, some number (say a) of sub-cubes or cubes would be visible with the slide touching the screen B—*i.e.* when $x=1$—and the relation would clearly be

$$y=ax^3,$$

each edge of the x-cubes having been divided into $\sqrt[3]{a}$ equal parts.

With the slide at any convenient mark (x) on the scale,

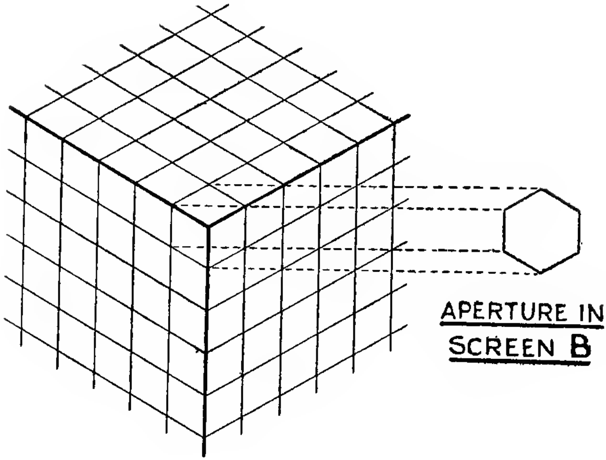

FIG. 83.

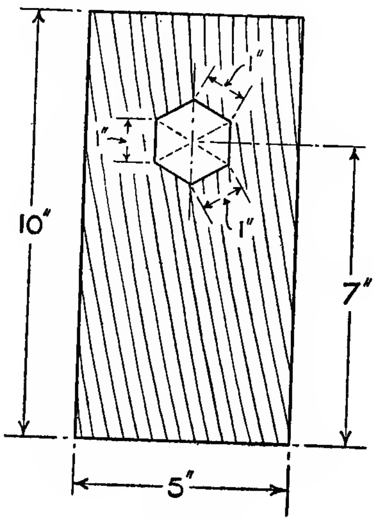

FIG. 84.

some number of cubes will be visible; and if the slide be then moved a small distance farther from the eye, x becoming $x+\delta x$, the visible cubes will be increased from y to $y+\delta y$.

The added cubes (or portions of cubes) will be arranged in three slab-like groups lying against three adjacent faces of the original cube-group. On closer examination, these slab-like groups will be found capable of arrangement in seven more convenient sub-groupings—viz. 3 slab-groups,

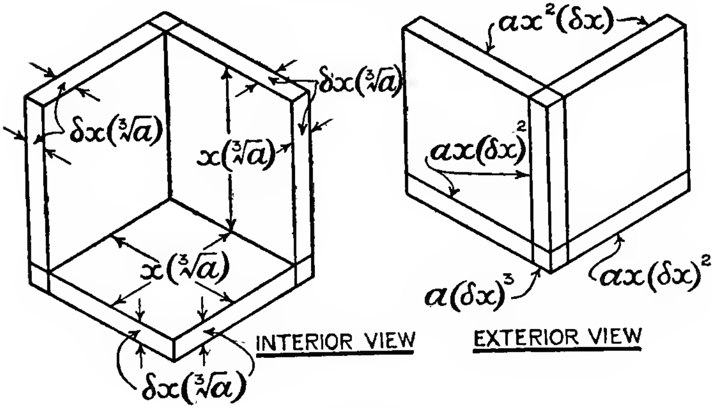

FIG. 85.

each $x(\sqrt[3]{a})\times x(\sqrt[3]{a})$ and $\delta x(\sqrt[3]{a})$ in thickness, the three together comprising $3ax^2\delta x$ things; 3 rod-like groups, each $\delta x(\sqrt[3]{a})\times\delta x(\sqrt[3]{a})$ and $x(\sqrt[3]{a})$ in length, these three together comprising $3ax(\delta x)^2$ things; and one small cube-group, $\delta x(\sqrt[3]{a})\times\delta x(\sqrt[3]{a})\times\delta x(\sqrt[3]{a})$, comprising $a(\delta x)^3$ things. These seven sub-groups are indicated, in their proper relative positions, in Fig. 85, division lines being omitted for the sake of clearness.

Hence

$$\delta y=3ax^2(\delta x)+3ax(\delta x)^2+a(\delta x)^3;$$

and since this increase in y is consequent upon an increase

δx in x, we may state that the rate of variation of y with respect to x, *for the specified range as a whole*, is

$$\frac{\delta y}{\delta x} = 3ax^2 + 3ax(\delta x) + a(\delta x)^2,$$

which would be true for that range as a whole, but not otherwise.

To determine the instantaneous rate of variation, we argue (as before) that if δx were made smaller than we could perceive, the rod-like groups, $ax(\delta x)^2$, would be yet smaller in comparison with the slab-groups, $ax^2(\delta x)$; and the corner cube-group, $a(\delta x)^3$, even smaller still. There is no need to discuss the relative smallnesses of these small things. It is sufficient to know that all four of them would be smaller than we could either perceive or conceive (though they have not been annihilated), and that, hence, they may be ignored.

The differential coefficient of $y = ax^3$ would therefore be

$$\frac{dy}{dx} = 3ax^2 \text{ things per unit change in } x\,;$$

but it is, of course, useless for calculation as a rate in the ordinary way, since it ceases to be true immediately any perceptible value, no matter how minute, is assigned to the change in x.

The integration is as simple as that for the quadratic variation. If we are told that $\frac{dy}{dx} = 3ax^2$, we are also told, by inference, that

$$\delta y = 3ax^2(\delta x) + 3ax(\delta x)^2 + a(\delta x)^3\,;$$

from which it follows that the variation is that of a cube-group with regard to the number of things lying along its edges, and hence that

$$y = \int 3ax^2(dx) = ax^3 + \mathrm{C}.$$

It would be quite easy—though perhaps somewhat tedious—to proceed through the cases of $y = ax^4$, $y = ax^5$,

and so on, until a general expression had been deduced for $\frac{d}{dx}(ax^n)$. There is, however, no need to do so if the work of Chapters III. and IV. has been thoroughly grasped; for it follows from this work that x^4, x^5 and the rest are but extensions of x^3 by simple repetition. The student is advised to consider a few special cases—such as $y=x^5$, $y=x^8$, $y=\sqrt{x}$, $y=\sqrt[3]{x}$, and $y=x^{\frac{3}{2}}$, with the aid of the apparatus of Fig. 76. The only modifications necessary are in the manner of preparing the slides, and the construction of the apertures in the screen B; and these are likely to prove more interesting, as well as more instructive, if left to the ingenuity of the student himself to devise. It is sufficient here to state that the apparatus will yield visual proof that, if $y=ax^n+C$, where n is any positive number, integral or fractional, then $\frac{dy}{dx}=nax^{(n-1)}$; and also that, if $\frac{dy}{dx}=nax^{(n-1)}$, then $y=\int nax^{(n-1)}\,.\,dx=ax^n+C$.

It is highly important to note that when we "integrate" a function, *we assume that the given function is a differential coefficient*, or instantaneous rate of change, relating to some variation between dependent groups of things; and the integration is permissible and real only so long as that condition is fulfilled. We are no more entitled to "integrate" an expression without knowing whether it actually represents a rate of variation or not than we are to write down a quadratic equation (see p. 183) without knowing whether it represents a real or realisable group of things.

To continue our consideration of $y=ax^n$, it will be clear that negative powers of x in this family must follow an order similar to that followed by positive powers; for (as we have seen) x^{-n} means simply $\left(\frac{1}{x^n}\right)$. And since x^n is merely a number of things, so also is x^{-n}.

A little consideration will show that x^{-n} is that fraction

of a group comprising x^n things which must be taken in order to obtain a single unit thing. Clearly, in the relation

$$y = ax^{-n},$$

an increase in x causes a decrease in y; and hence the variation is "inverse." The differential coefficient is therefore negative—signifying merely that a change in x produces a change of the opposite sense in y.

Hence, if $y = ax^{-n}$, it follows that $\frac{dy}{dx} = -nax^{(-n-1)}$; and given that the instantaneous rate of variation between two groups of things is $\frac{dy}{dx} = -nax^{(-n-1)} = \left\{ -\frac{na}{x^{(n+1)}} \right\}$, it follows that the variation is of the type

$$y = \int -nax^{(-n-1)} \,.\, dx = ax^{-n} + \mathrm{C}.$$

Moreover, this variation may be visualised with the apparatus of Fig. 76 just as well as the preceding cases—only saying what fraction 1 thing is of the visible groups, instead of dealing directly with the numbers of things in the visible groups. The student should obtain the necessary data, by observations with the apparatus, for a few typical cases, such as $y = ax^{-3}$, and $y = ax^{-\frac{1}{2}}$, both for differentiation and integration.

The Trigonometrical Functions.—The differentiation and integration of the trigonometrical functions—Sin θ, Cos θ, and Tan θ—with respect to θ may be seen from inspection of the ordinary diagram, so simple and obvious are they.

Consider Fig. 86, in which are shown an angle θ and a small increment-angle $\delta\theta$, with the lines employed for the measurement of these angles.

Now, if the length OA (the radius of the circular arc) be regarded as a unit distance, the fraction of such a unit in BD will be equal to Sin θ; CE will be equal to Sin $(\theta + \delta\theta)$; OD = Cos θ; OE = Cos $(\theta + \delta\theta)$; AF = Tan θ; and AG = Tan $(\theta + \delta\theta)$. Similarly, the number of such units (or fraction of such a unit) in the arc AB will be equal to

the number of radians (or fraction of a radian) in the angle θ; and the number in the arc AC to the angle $(\theta + \delta\theta)$.

If θ be increased to $(\theta + \delta\theta)$—*i.e.* from AB to AC radians—the Sine will be increased from BD to CE, the increment in the Sine being PC, since BP is parallel with OA; and the causing increment in the angle is BC.

Obviously, then, the rate at which the Sine varies with

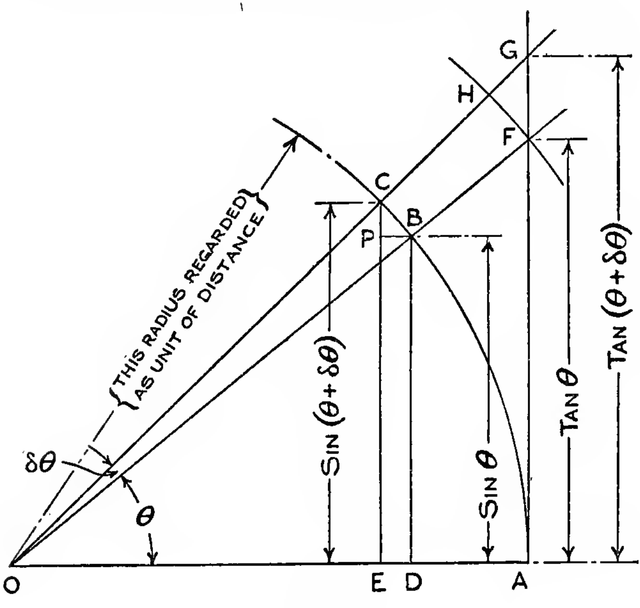

FIG. 86.

respect to the angle is $\frac{\delta}{\delta\theta}(\sin\,\theta) = \frac{\text{PC}}{\text{BC}}$; and if $\delta\theta$ (and hence BC) were made smaller than we could perceive, there would be no appreciable curvature in BC, the departure of the chord BC from the arc BC being imperceptible. The line BC would then be sensibly straight, and at right angles to OB; and the triangle PCB would be similar to the triangle BOD—or, more correctly, these triangles would

then be similar within a "limit of tolerance" smaller than we could perceive. Therefore

$$\frac{d}{d\theta}(\sin\theta)=\frac{PC}{BC}=\frac{OD}{OB}=\cos\theta.$$

What this differentiation really amounts to is this. Granted unit radius for the arc, θ is measured by the length of the *arc*, and Sin θ by the length of the *perpendicular*. Comparing these two ratios, therefore (the consequent of each being the unit distance $OA=OB=OC$), we do but compare the number of length units involved in a *change in the length of the perpendicular* with those involved in the *corresponding change in the length of the arc*. It is evident from the diagram of Fig. 86 that as the change in the angle is made smaller and smaller, the ratio borne by the first of these differences to the second becomes more and more nearly equal to the ratio $OD : OB=\cos\theta$.

Conversely, if we are told that the instantaneous rate at which one group of things varies with respect to another group of things is $\frac{dy}{d\theta}=\cos\theta$, we are told also, by inference, that the variation is either identical with or similar to that of the Sine with respect to the circular measure of an angle; and hence that

$$y=\int\cos\theta\,.\,d\theta=\sin\theta+C.$$

The constant of integration is inserted to allow for the assemblage comprising a number of things (all similar to those represented by the term Sin θ, of course) which are so placed that they are in no way affected by change in the angle θ.

With $y=\text{Cos}\,\theta$, an *increment* $\delta\theta$ in the angle causes a *decrement* in the Cosine; and hence

$$\frac{\delta}{\delta\theta}(\cos\theta)=-\frac{BP}{BC},$$

which, if $\delta\theta$ were made smaller than we could perceive, would become

$$\frac{d}{d\theta}(\cos\theta) = -\frac{BP}{BC} = -\frac{BD}{OB} = -\sin\theta.$$

Given this as an instantaneous rate of variation, we know that the variation must be that of the Cosine with respect to the circular measure of an angle; and hence

$$y = \int -\sin\theta \,.\, d\theta = \cos\theta + C\,;$$

or, given that the instantaneous rate of variation is equal to Sin θ,

$$y = \int \sin\theta \,.\, d\theta = -\cos\theta + C.$$

In the relation $y = \text{Tan}\ \theta$, an increment $\delta\theta$ in the angle causes an increment FG in the tangent; and hence

$$\frac{\delta}{\delta\theta}(\tan\theta) = \frac{FG}{BC}.$$

Now, these two lines do not intersect; and they are therefore useless as they stand. They may, however, be made to serve our purpose by drawing the circular arc FH with centre O.

The arc FH = the arc BC increased in the ratio OF : OB, which (since OB and OA are radii of the same arc) is equal to the ratio OF : OA; and this ratio we call Sec θ.

Hence

$$\text{arc FH} = \text{arc BC} \times \sec\theta,$$

and

$$\text{arc BC} = \text{arc FH} \times \cos\theta.$$

Therefore

$$\frac{\delta}{\delta\theta}(\tan\theta) = \frac{FG}{BC} = \frac{FG}{FH\,.\,\cos\theta} = \frac{FG}{FH}\,.\,(\sec\theta).$$

If the increment-angle were made smaller than we could perceive, so also would be the arc BC. The arc FH would become, sensibly, a straight line perpendicular to OB; and the difference between the angles HFG and BOD would be imperceptible. Hence, HFG and FOA would be, in effect, two similar triangles; and the ratio FG : FH would be equal to the ratio OF : OA = sec θ.

Therefore

$$\frac{d}{d\theta}(\tan\theta) = \sec\theta \times \sec\theta = \sec^2\theta.$$

Conversely, given Sec² θ as an instantaneous rate of change, we know that the variation must be that of the tangent with respect to the circular measure of an angle—*i.e.* of the length of the tangent with respect to the length of the arc—so that

$$y = \int \sec^2\theta \,.\, d\theta = \tan\theta + \mathrm{C}.$$

Much useful information may be obtained by plotting the curves for Sin θ and Cos θ with regard to ordinary rectangular axes (showing the variations in these ratios with respect to variations in the angle θ), and examining the two curves in their relation to each other.

The curves are shown for $\theta = 0$ to 2π in Fig. 87; and it will be seen that the ordinates of the Cosine curve show the "slope" of the Sine curve—*i.e.* the instantaneous rate at which the Sine varies with respect to the arc—for all angles. Also, the ordinates of the Sine curve show the integral of the Cosine curve.

In this contemplation of the curves, it must be carefully borne in mind that they are merely *diagrammatic representations* of real things; and any deduction or inference drawn must be interpreted and tested by reference to the real things concerned, instead of to their diagrammatic representations. The point at issue, and its significance, will perhaps be seen more clearly from the consideration of a typical example:

$$y = \int_{\theta=\frac{\pi}{3}}^{\theta=\frac{\pi}{2}} \cos\theta \,.\, d\theta = \left(\sin\frac{\pi}{2} + \mathrm{C}\right) - \left(\sin\frac{\pi}{3} + \mathrm{C}\right)$$
$$= \sin\frac{\pi}{2} - \sin\frac{\pi}{3}.$$

This is usually interpreted as meaning that "y is the *area* under the Cosine curve between the ordinates for $\theta = \frac{\pi}{3}$ and $\theta = \frac{\pi}{2}$," shown hatched in Fig. 88; the *area* being

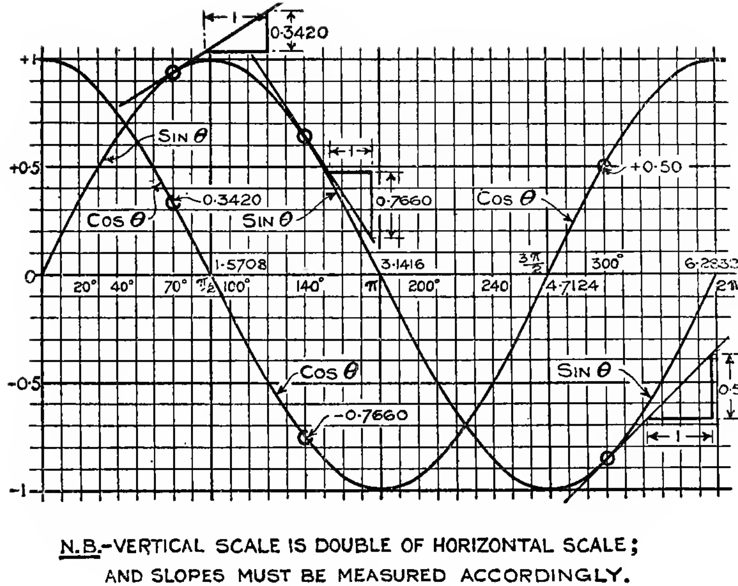
0·3420
Sin θ
Cos θ
0·3420
Sin θ
0·7660
Cos θ
+0·50
Cos θ
Sin θ
0·50
−0·7660
1·5708
3·1416
300°
6·2832
20°
40°
70°
π/2
100°
140°
π
200°
240
3π/2
4·7124
2π
+1
+0·5
0
−0·5
−1
N.B.—VERTICAL SCALE IS DOUBLE OF HORIZONTAL SCALE;
AND SLOPES MUST BE MEASURED ACCORDINGLY.

FIG. 87.

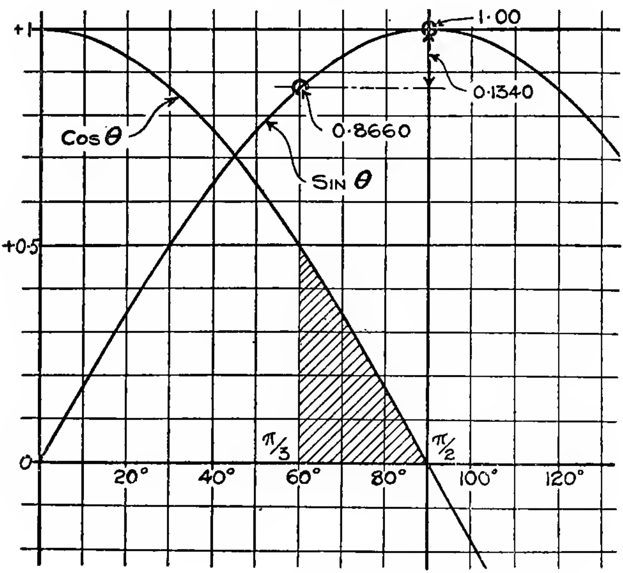
1·00
0·1340
0·8660
Cos θ
Sin θ
π/3
π/2
+1
+0·5
0
20°
40°
60°
80°
100°
120°

FIG. 88.

taken as the *product of the angle* $\left(\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}\right)$ *with the average value of the Cosine over that range.*

Now, it is safe to say that such "explanations" have never been accepted as satisfactory by any intelligent student. They are often taken on trust for the sake of peace and quietness—and perhaps even more often they are "learned" parrot-fashion because one may thus acquire credit for knowledge more easily than by demanding a plain, common-sense interpretation in the light of simple fact—but it is quite certain that they have never helped any man to a real, true comprehension of mathematics for practical purposes.

Let us see what is the interpretation and significance of such an expression.

The given statement $y=\int_{\theta=\frac{\pi}{3}}^{\theta=\frac{\pi}{2}}\cos\theta\,.\,d\theta$ implies that two groups of things denoted as y and θ are varying dependently; and also that the instantaneous rate of variation between them is

$$\frac{dy}{d\theta}=\cos\theta.$$

This brings us at once to the diagram of Fig. 86; and as the angle θ changes from $\frac{\pi}{3}$ (*i.e.* 60 degrees) to $\frac{\pi}{2}$ (*i.e.* 90 degrees), the variation in y will be as shown, for the sake of clearness and to facilitate reference, in Fig. 89.

For the rate of variation to be PC : BC, it is obviously essential that the increment PC shall accrue to a *perpendicular*, since the basic principle of mathematical grouping demands that all things forming an assemblage shall be sufficiently similar to permit of their being counted under a single denomination. Clearly, then, the group y, to which the increment dy is added, must be the perpendicular BD, which, with the radius OA equal to one unit of distance,

measures by its length the Sine of θ. The addition of PC to BD brings the perpendicular to CE, of course; but it is still the perpendicular, measuring the Sine of the angle $(\theta + d\theta)$.

The total addition to BD (*i.e.* to y) accruing while θ changes from 60° to 90° (*i.e.* from $\frac{\pi}{3}$ to $\frac{\pi}{2}$) is, therefore, a perpendicular line made up of an infinitely large number

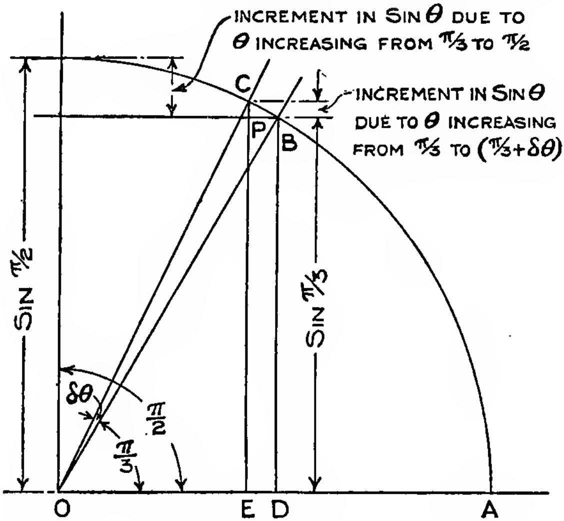

FIG. 89.

of infinitely small lengths such as PC; the ultimate length of the perpendicular measuring Sin $\frac{\pi}{2}$, just as its initial length measured Sin $\frac{\pi}{3}$.

Since $\frac{dy}{d\theta} = \cos\theta$ is only an *instantaneous* rate of change, we cannot obtain an accurate determination of the total increment of y over the given range by supposing the rate Cos θ to apply over any perceptible range, no matter how

minute; for if we assign to the increment angle $\delta\theta$ any perceptible magnitude, there is at once a perceptible difference between the arc BC and the chord BC. Just as, with $\frac{dy}{dx}=3x^2$, we knew that there were other terms which must be taken into account if any magnitude, no matter how minute, be assigned to dx (or, more properly, to δx, since the increment would then have become perceptible), so with $\frac{dy}{d\theta}=\cos\theta$ we know that there are other terms (concerning the difference between the arc BC and the chord BC) which must be taken into account if even the smallest conceivable magnitude be assigned to the increment in θ. Moreover, just as we avoided the trouble of writing those other terms—$3x(\delta x)^2$, etc.—when dealing with $\frac{dy}{dx}=3x^2$, by noting and agreeing that the term $3x^2(\delta x)$ indicated three slab-like increment groups fitting upon three adjacent faces of a main cube-group, and that hence the variation must be that of $y=x^3$, so we avoid the trouble of writing these other "curvature" terms when treating $\frac{dy}{d\theta}=\cos\theta$, by noting and agreeing that they are implied in the given statement just as definitely and imperatively as though they were written.

This constitutes a vitally important point in the principle of the Calculus—viz. that if we write δx for the change in x, we are understood to mean a change of some perceptible or conceivable magnitude; and δy (the consequent or corresponding change in y) must then account for *all component parts of that change*. If, on the other hand, we write dx for the change in x, we are understood to mean that the value of dy stated is only so much of the whole change in y as we could perceive if the increment in x were so small as to be both imperceptible and inconceivable in its smallness. Hence, we are allowed to omit the very small terms *from the written expression for the*

instantaneous rate of change (and from that alone) on the distinct and essential understanding that we advertise the fact of their existence, though of inconceivable smallness, by writing dx and dy instead of δx and δy for the changes in x and y. The fact of the existence of these terms is by no means less important than their smallness.

An approximate determination of $y=\int_{\theta=\frac{\pi}{3}}^{\theta=\frac{\pi}{2}} \cos\theta \, . \, d\theta$ may be obtained, for amplification of Fig. 89, by supposing the instantaneous rate of change (Cos θ) to hold over small ranges of angular change. Thus, noticing that the total change in θ is $\frac{\pi}{6}=30$ degrees, we might divide this into 5 ranges of 6 degrees each, and assume the average value of Cos θ through each of those ranges to be the arithmetic mean of its values at the extremities of the ranges. The estimate might be set out as shown on p. 268.

From the trigonometrical tables, Sin $90^\circ=1$ and Sin $60^\circ=0{\cdot}8660$; whence Sin 90° − Sin $60^\circ=1-0{\cdot}8660=0{\cdot}1340$, and the discrepancy between this and the estimated $0{\cdot}133841$ is due to disregard of the "curvature" terms implied in the statement $\frac{dy}{d\theta}=\text{Cos}\,\theta$. A closer approximation could be obtained by dividing the angular change into a larger number of smaller ranges; and this may be left to the student as an exercise, as may also the cases of

$$y=\int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos\theta \, . \, d\theta=\sin\frac{\pi}{2}=1,$$

and

$$y=\int_{\theta=0}^{\theta=\frac{\pi}{3}} \cos\theta \, . \, d\theta=\sin\frac{\pi}{3}=0{\cdot}8660.$$

Change in θ (θ_1 to θ_2).	$\delta\theta$ in degrees.	Cos θ_1.	Cos θ_2.	Assumed average magnitudes of $\cos\theta = \left(\frac{\cos\theta_1 + \cos\theta_2}{2}\right)$.	Range of assumed applicability $=\delta\theta$ in radians.	Increment in $y=$	$\delta \sin\theta$.
60° to 66°	6	0·5000	0·4067	0·45335	0·10472	0·45335 × 0·10472	0·047474
66° to 72°	6	0·4067	0·3090	0·35785	0·10472	0·35785 × 0·10472	0·037473
72° to 78°	6	0·3090	0·2079	0·25845	0·10472	0·25845 × 0·10472	0·027065
78° to 84°	6	0·2079	0·1045	0·15620	0·10472	0·15620 × 0·10472	0·016357
84° to 90°	6	0·1045	0	0·05225	0·10472	0·05225 × 0·10472	0·005472
						Total .	0·133841

The great necessity is, however, for a full and clear understanding of the realities which are implied and expressed by the various statements; and a grasp of those realities so firm and sure that while the hand is writing symbols and expressions, the mind sees, without the slightest vagueness, the actual realities undergoing the changes represented by the expressions.

In the mind of the thoughtful student there may arise some such question as this: "You say that, given $\frac{dy}{dx}=3x^2$, you *know* the variation is that of x^3 with respect to x; but surely this assumes that when the $3x^2(\delta x)$ slab-groups have been fitted against three adjacent faces of a cube-group, other subgroups will fit in to complete the cube-group. How would it be if they did not so fit; and how is one to be sure? Again, given $\frac{dy}{d\theta}=\text{Cos }\theta$, you assume that there are other terms (referred to as *curvature* terms) implied, which justify the inference that the variation is that of the Sine with respect to the Circular Measure of an angle θ. How if there were no such curvature terms; or if there were other terms which would not fit properly for that inference?"

This is a question well worth asking; and well worth pondering, even after it has been answered with seeming satisfaction.

The question really amounts to this: "*Can there be two different kinds of variations which would give the same differential coefficient?*" And the answer is No! The statement $\frac{dy}{dx}=3x^2$ is a complete specification of a cube-group variation; and the statement $\frac{dy}{d\theta}=\text{Cos }\theta$ stipulates uncompromisingly that other "curvature" terms shall be reckoned with in integration, those terms being such as will give $y=\int \text{Cos }\theta \,.\, d\theta=\text{Sin }\theta+\text{C}$, and that alone.

One instance stands out, and sometimes gives trouble

through being wrongly regarded as an exceptional case—a sort of black sheep—of the x^n family. This instance is $\frac{dy}{dx}=\frac{1}{x}=x^{-1}$; which, on application of the ordinary "rule" for the integration of x^n, gives

$$y=\int x^{-1}\,.\,dx=x^0\div 0,$$

and this, of course, is unintelligible.

The fault lies, however, in a misapplication of the rule—and many things are *applied* which have no true *applicability*, the results being always regrettable, if not worse.

We may apply to x^n functions the operations which are properly applicable to them; but not others. There is no x^n function which gives, on differentiation with respect to x, a differential coefficient $\frac{dy}{dx}=x^{-1}$; and hence the rule for the integration of x^n is not applicable.

If $y=ax^2$, then $\frac{dy}{dx}=2ax^1$;

If $y=ax^1$, then $\frac{dy}{dx}=ax^0=a$;

If $y=ax^0$, then $\frac{dy}{dx}=0\times ax^{-1}=0$; and

If $y=ax^{-1}$, then $\frac{dy}{dx}=-1\times ax^{-2}=-ax^{-2}$.

This case, therefore, is not an exceptional instance in the family of x^n functions; for it does not properly belong to that family at all.

The differential coefficient $\frac{dy}{dx}=\frac{1}{x}=x^{-1}$ is obtained from the differentiation of the relation $y=\log_\epsilon x$ with respect to x; and hence it follows that, given $\frac{dy}{dx}=x^{-1}$—

$$y=\int x^{-1}\,.\,dx=\log_\epsilon x+\mathrm{C}.$$

Functions in general.—A good deal of vagueness and mystification arises from a loose employment of the term

"function." The word function means something in the nature of *working* or *operation* ; and hence an expression such as

$$y = x^4 - 7x^3 + 3x^2 - 11x + 5$$

may well be termed a "function of x," since it represents an assemblage of things gathered together by repetition of an original unit thing, through the *operation* of x as a factor of repetition.

Similarly, the relation

$$y = \sin \theta$$

may well be termed a "function of θ," since it represents an assemblage of things gathered together in consequence of the *operation* of θ.

It is often convenient to use the word "function" as a means for saving time and trouble in writing and speaking ; but there can be no question that it is better always to *think* of such expressions as $y = x^4 - 7x^3 + 3x^2 - 11x + 5$, and $y = \text{Sin}\ \theta$, as representing *numbers of real things forming assemblages* which are to be counted, or compared numerically, and to visualise the implied assemblages by means of convenient symbols, either actually or mentally, according to the needs and conveniences of particular cases.

CHAPTER XII

UNITS, FUNDAMENTAL AND DERIVED

Concepts and Percepts.—There is a good deal of vagueness regarding "concepts" in engineering science; and as such vagueness is opposed to the firm grasp and clear vision which it is here sought to foster, the matter is worth consideration.

So far from a desire to promote discussion concerning mere words and their meanings, it will be seen that the object in directing attention to this point is to remove ambiguity by showing that it is at least open to question as to whether the word "concept" need be used at all in this connection; and that, if it be used, its meaning is really quite simple and straightforward.

A "concept" is an *idea*—something conceived within the mind, and altogether apart from external objects. A "percept," on the other hand, is a mental impression, produced by some external object through the operation of one or more of the senses.

It would be difficult to specify the limitations and scope of ideas, for a fertile imagination may produce them with the utmost facility, and in great volume over a very wide range; but it is obvious that the boundary which divides the useful concept from the phantasm of a midsummer night's dream is very narrow.

From the earliest times, men have conceived ideas in their search after solutions for their difficulties and problems; and many valuable lessons may be learned from a considera-

tion of these ideas with regard to their foundations and consequences.

For instance, the idea of a composite creature possessing the intelligence of a man in combination with the strength and speed of a horse was, perhaps, a not unnatural conception for a man whose power of fantastic imagination was superior to his skill in horsemanship; but it is as far from fulfilment as ever—though the alternative idea of producing a *machine* which should combine greater power and speed than those of a horse with complete submission to the will of a man has been realised in large measure, and has led to important consequences.

Now, the self-propelling car in general is but an application of things which were evidently *observed* by men from natural happenings. A round stone would roll down a slope more readily than would one with broad faces upon it; and this, doubtless, led to the wheel as a means for reducing the resistance to locomotion. The idea of a wheel propelled by wings similar to those of a bird was, of course, totally impracticable—but the idea of devising a machine which would cause a wheel to rotate and roll along a level surface, and even up an incline, has been fruitful in many ways.

Similarly, those who sought to overcome the obstacles presented by seas, lakes and rivers through the use of artificial propelling attachments resembling those possessed by fish were doomed to failure; as also were those who strove to acquire mastery of the air by the use of wings like those of birds. It remained for others to find, by closer observation, that a fish moves through still water, and a bird soars and flies in still air, by exerting pressure upon the surrounding medium; that the tail and fins of a fish, like the wings of a bird, must be directly and entirely under the control—through systems of nerves and muscles acting together—of a brain which has evolved with them; and that, even could he imitate them perfectly, the movements which serve to propel a fish through the water, or

those by which a bird travels through the air, would be useless to a man because of his essentially different construction and development.

Countless other instances might be quoted; but this is not a suitable occasion for an exhaustive discussion of such matters—though their importance and interest should be sufficiently obvious to attract the attention of every serious student of engineering, with a view to discovering the true reasons for the fact that some ideas are practicable while others are not. It must suffice here to suggest that, as regards practical utility, the "percept" would appear to be far more important than the "concept"—which means that, in engineering, the ability to see things as they really are, without distortion, and to describe them as they actually appear, without exaggeration, belittlement or mysticism, is more important than the faculty for imagining strange things as they never were and probably never will be.

Of course, generalisation is always more or less dangerous, and the foregoing must be read as a mere suggestion of tendency rather than as a dogmatic assertion. In a world where nothing is ever reproduced exactly, and where every state is but a passing phase—a world of which the only constant property is changefulness, and where the fancies of yesterday are the facts of to-day, while that which appears to be unassailable truth to-day may stand exposed as falsehood to-morrow—our knowledge is (and must always be) incomplete, and largely mixed with mere assumption or speculation. For this very reason, however, it is all the more desirable that our views and actions should be as simple and natural as possible, based upon, and in accordance with, perception, rather than conception, and as free as may be from any taint of mysticism used with the object of acquiring credit for the possession of supernatural powers.

It is important to notice that, where we cannot perceive the fundamentals, our measurements—*i.e.* our compari-

sons regarding magnitudes—are based always upon perceptible and perceived effects, and not upon conceived ideas.

For example, we cannot observe force; and since we cannot perceive it, neither can we form any reliable conception of it. We are only aware of its action through its perceptible effects upon bodies. We say that a wind is blowing when we see trees bending before it, hear the sounds caused by the vibrations set up in the air, or feel it assisting or hindering our movements. Forces of enormous magnitude might (and doubtless do) pass close to us frequently without our being aware of their existence—simply because they do not act upon bodies with effects which we may perceive. In comparing "forces," a force of 5 lb. means to us merely something which, if applied to a body, will set up five times as much motion-producing tendency as would be set up by our conventional 1-lb. weight. Moreover, the commonly accepted statement to the effect that "a force may be represented in magnitude and direction by a straight line" should be examined with the closest care, and interpreted with a proper regard to the ascertained facts; for that which we represent by a straight line in "force-diagrams" is really neither more nor less than the path which would be taken, and the amount of motion which would be acquired, by a body under the action of the specified force.

These matters are commended to the student as warranting his most careful and searching investigation. He should endeavour to realise the fact that our measurements are based entirely upon perception; and that "concepts," unless they be based upon, and in accordance with, "percepts," are liable to hinder instead of assisting practical accomplishment. In all measurement, the thing to be measured must have been perceived—otherwise there would be no need to measure it—and all that we have to do is to arrange a convenient and properly acceptable *unit* which may serve as a basis for comparison.

Number.—Number is the result of counting real things in groups; and apart from this, figures and symbols have no numerical significance. We might, for instance, brand a truck with the figure 5; but unless four other trucks had preceded it, or could be in some way grouped with it so as to properly bear the figures 1, 2, 3 and 4, the 5 would be meaningless except as a mere distinguishing mark without reference to number—and in such case the truck might as well have been branded with a □, ○, △ or any other of the distinguishing marks which have been in use for untold centuries.

We may count real things in whole numbers from 1 upwards, and in fractional parts; while, as shown in the preceding pages, the unit for counting may be a single whole thing, an assemblage of whole things, a fraction or part of a whole thing or an assemblage of fractional parts.

As shown in Chapter II., there is no such thing as a *negative* number. A number of things may be regarded as "positive" when they have the effect of either increasing the assets or reducing the liabilities of some group; and as "negative" when they have the effect of either reducing the assets or increasing the liabilities of a group. In transactions and operations where groups of things pass from one ownership to another, the self-same things which increase the assets of one person reduce those of another; and hence, all numbers of things in such cases are simultaneously positive and negative, according to the standpoint from which they are regarded—but the things themselves are real and unchanged all the time.

Point.—A point is some *thing* smaller than we have either the need or the means for measuring. Points are, however, always three-dimensional; otherwise they would be not only of a smallness, but also of a nature, inconceivable to human beings.

One can feel with Euclid in his definition of a point as having position but not magnitude. There were then, as there are now, people who were quibblers by construction

—people who saw in a statement nothing but a chance to embarrass its author by the mere wording of the statement. These would have asked whether the distance " between " two points would be measured over-all, from centre to centre, in-to-out, or on what other basis; ignoring the fact that they had no instruments of sufficient delicacy to record any difference between measurements made on the most widely divergent bases, even were the points so large as one-thousandth of an inch across. It is typical of such people that they succeed in combining a very low standard of personal attainment with a tremendously high ideal, just as another class combine an extremely low standard of personal conduct in themselves with a most austere and rigid code of morals for others.

As our skill improves, we are able to make instruments to deal with smaller and smaller measurements; and the " point " becomes smaller as we advance—but it is always *real*, no matter how small. It is a point of steel, of wood, stone, clay, brick, air, water or anything else, according to the circumstances. We cannot reduce a piece of steel to such a smallness that it ceases to be steel—at least in practical engineering work where we are concerned with the comparison of magnitudes. In that field, the smallness of a body does not affect its nature.

Line.—Here, again, it is probable that the same difficulty prompted Euclid to define a line as possessing length but not breadth. Others, seeking a higher degree of abstraction, speak of a line as the " path of a moving point." It is somewhat unfortunate for the propounders of this latter view that Euclid was so severe as regards the point; for if it has no magnitude, it certainly cannot contain propelling machinery, nor can it possess any means whereby an external force may be applied to make it move.

A line is merely an assemblage of contiguous points. Hence, a line is three-dimensional; but as its breadth and thickness do not concern us, we ignore them, concentrating attention upon its length alone.

How many points are there in the diameter of a halfpenny? We do not know; nor do we know even that all points are of uniform size—indeed, there is good reason to believe the contrary, since our grandfathers would have considered "inappreciably small" what is now regarded as sufficient to warrant the condemnation and rejection of an ordinary commercial machine-part. We do not know; and we do not care. We leave number alone, and say that there are *one inch* of them.

Definitions for the various kinds of line—straight, curved, helical, etc.—follow simply on this basis.

Care should be taken to avoid confusion of "a line" and "length." The line is a real thing; and is not in any way affected by being (or not being) measured. Length is merely measurement of the line—*i.e.* a comparison of it with some conventional unit of distance.

Surface.—A plane surface is made up of parallel lines in contact. Surfaces other than plane surfaces may or may not be made up of straight lines—for instance, a cylindrical, a conical and a helical surface contain lines which are straight as well as lines which are not straight; while a spherical surface contains no lines which are straight.

A surface, whether plane or otherwise, is three-dimensional; but we ignore its thickness when that dimension is not germane to our purpose.

How many lines, each one inch in length, must be placed side by side to make a width of, say, one inch? We do not know; and we do not need to place them, since they are always placed for us through our manipulation of material. Here, again, we leave number alone, and say that there are *one inch* of them, giving an "area" of one square inch.

Obviously, a surface is distinct from an area, just as is a line from a length.

Solid.—A solid is made up of parallel surfaces in contact.

There is no need for further discussion here, except to

point out that a solid is a real thing, unaffected by measurement; while volume is merely an expression of measurement—*i.e.* of comparison with some accepted unit.

A point arises in connection with the mensuration of solids which are said to be "generated" by a moving surface, vagueness and confusion often being introduced where the facts are perfectly plain and simple if not obscured by unnecessary rules and formulae. An instance of this is to be found in the Theorems of Guldinus, the first of which is commonly stated in some such form as the following: "The volume generated by the revolution of a plane figure about any external axis in its plane is equal to the product of the area and the distance moved through by the centre of gravity of the area."

Now, such solids may be quite easily treated for the determination of their volumes by taking them as they are, with just so much adaptation as is necessary to permit the statement of their breadth, thickness and average (or effective) length.

Consider such a typical case as the anchor-ring; and think for a moment of a piece of it lying between two cross-sections radiating from the centre of the ring—such as AB and CD in Fig. 90. This piece will be what might be regarded as a cylinder with a curved axis.

If the sections AB and CD were brought (by some power higher than ours) so close together that the distance between them were smaller than we could perceive, the curvature of the axis—*i.e.* its departure from a straight line perpendicular to the section-surfaces—would be imperceptible to us; and we might imagine the effect of taking a number of such discs, and placing them to form a vertical pile, with alternate discs reversed, as shown (much magnified for the sake of clear illustration) in Fig. 91.

The discs would still be thicker at one side than at the other, of course, though the difference would be imperceptible to us; and hence we must deal with their true *average* or effective thicknesses, measured on the axis

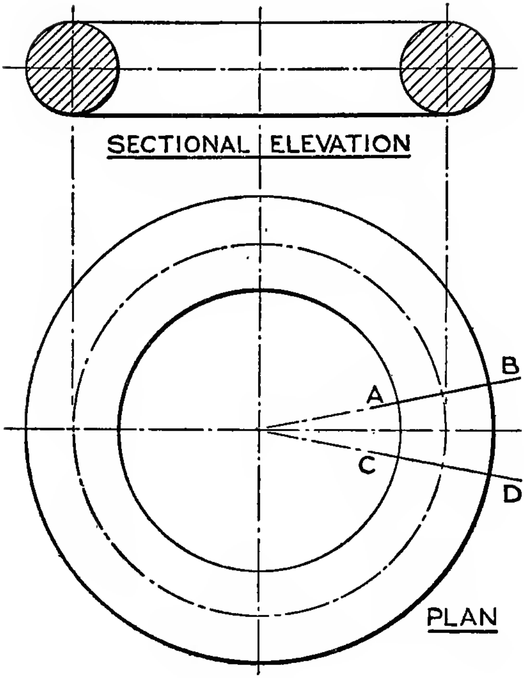

FIG. 90.

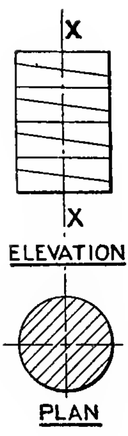

FIG. 91.

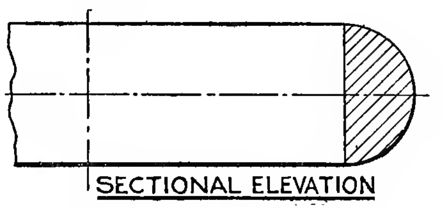

FIG. 92.

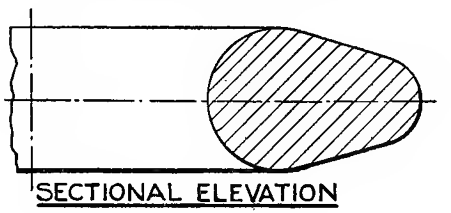

FIG. 93.

XX. Each of these discs has a volume, and since we know the area of the section-surface, we may argue that if a sufficient number of such extremely thin discs were piled up to make the axis one inch in length (or height), there would be as many cubic inches in the volume as there are square inches in the area of the section-surface. This gives a group of cubic inches; and it is obvious that the complete solid will contain as many such groups as there are inches of length in its axis—and we must deal, as accurately as we are able, with the actual length of the axis.

It should be noticed that, no matter how thin the discs might be, if they were piled to a height of one inch, in their proper relative positions, the axis would be curved. The benefit to be derived from imagining them infinitely thin, and the alternate discs reversed, is simply that we are thereby enabled to locate and deal with the *axis of effective thickness*. With a circular, square or other completely symmetrical section, this axis of effective thickness is halfway across; but if the section were a semicircle (as in Fig. 92), or some unsymmetrical shape considered specially suitable for ring construction (such as that shown in Fig. 93), the axis would not be halfway across between the inner and outer extremities. It would be at the centre of gravity of the section, which must then be carefully located.

Motion.—A body is said to have "moved" when its position has changed with regard to some other body which, for the particular purpose in view, is considered as stationary or at rest. Similarly, a body is said to be "moving" or "in motion" when its position is changing with relation to some other body which, for the particular purpose in view, is regarded as stationary. Motion, therefore, is change of position with relation to some point which is regarded as fixed.

Sometimes a certain amount of difficulty and confusion is caused through lack of precision in defining the basis upon which motions and movements are measured or properly measurable.

In stating the distances travelled in certain periods of time by a train which leaves one town on a journey to another, the points regarded as fixed would be points at the stations of departure, of destination and along the route; and distances would be measured along the track —though this might generally be far from a straight line over any appreciable part of the journey.

For the bulk of practical cases, this method serves and is employed; the path—whether straight or otherwise—of the moving body passing through the point which is regarded as fixed. No trouble arises in such circumstances, of course; but where the path of the moving body does not pass through the stationary point, distances between the body and point cannot properly be measured along the path of the former.

The essential difference between the two cases lies in the fact that where the fixed point lies in the path of the moving body, the latter will, sooner or later, arrive at the fixed point, provided the motion be continued sufficiently; whereas a body moving along the straight path PC in Fig. 94 will never arrive at the point O, no matter how long the motion be continued, nor how far the body move—indeed, after passing the point in its path at which its distance from O is a minimum, the farther it moves, the more will it recede from O—unless the path be suitably changed.

It will be clear that the only proper basis for measurement of the distances between the moving body and the fixed point O in Fig. 94, as the body moves along the path PC, is to measure the actual distances between the body and the point at the various stages of the motion, stating also the "bearing" of the body from the fixed point with regard to some basis of direction. Thus, at A, the distance of the body from O is the length OA, at the angle AOX with the axis XX. Similarly, at B, its distance is OB, at the angle BOX; and the distance is a minimum at C, the angle OCP being a right angle. This method is

an application of that known as "polar co-ordinates," and will need no further discussion here.

The positions of the body with relation to O might, of course, be specified by its co-ordinates referred to the axes XOX and YOY (which are shown as rectangular in Fig. 94, this being the most convenient method for practical work) or any other pair of axes; and the equation of the path could then be readily determined. The distances OA, OB, etc., would then be each the hypotenuse of a right-angled triangle—OAX_A, OBX_B, etc.—of which the sides about the right angle are known.

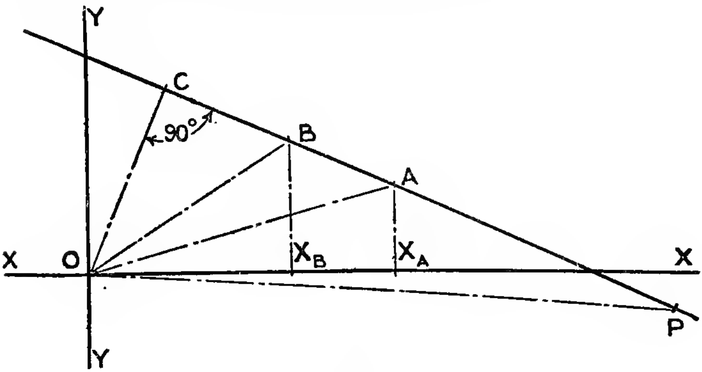

FIG. 94.

Such cases do not (or, at least, need not) often arise in practical work—valve motions and similar mechanisms, which provide typical instances of them, submitting usually to treatment by other methods; but clarity of view regarding motion and its measurement in general is of such extreme importance that it should be considered carefully and thoroughly, in all its aspects, from the standpoint of actual fact.

Velocity.—Velocity is the time-rate of motion—*i.e.* it is the rate at which the position of a moving body changes with respect to time.

A cricket ball may be bowled so that it travels the

length of the pitch in 1·5 seconds; and on the assumption that its velocity was constant (or, alternatively, considering the movement through the 66 ft. as a whole, without regard to changes of velocity within that range), we might say that its velocity was 44 ft. per second. It would be quite permissible to state this also as 30 miles per hour, although it is quite obvious that the ball as bowled would not travel 30 miles, nor continue in motion for an hour. What we mean is that the ball did actually move, and the *rate* at which it travelled was such that, had it continued (or did any other body continue) in motion at the same rate for an hour, it would have travelled 30 miles.

It is important to notice that a velocity is essentially a *distance*; and the object of its determination or specification is that we may know the *total distance* through which a body moving with stated velocity will travel in a certain period of time if its rate of motion be maintained throughout that period. The time factor is, of course, highly important, and must not be in any way disregarded or tampered with; but the distance factor is by no means less important, as will be seen on consideration.

Care is necessary to avoid loose thinking in all matters concerning rates. Velocity is *not* the quotient obtained from dividing distance by time—an operation which is, clearly, a physical impossibility. When we say that since a body moved with uniform motion through a distance of 100 ft. in 5 seconds, its velocity was therefore 20 ft. per second, we do not divide 100 ft. by 5 seconds. All that we do is, on the strength of the statement that the motion was uniform over the range of 100 ft. actually covered, to divide the 100 ft. into as many equal sub-distances as there were seconds in the period taken, allocating one of the equal sub-distances to each of the seconds.

From this it follows that velocity is the differential coefficient of distance with respect to time concerning the motion of a moving body; and where the movement is

not uniform, the methods of differentiation must be employed to determine velocity.

Acceleration.—Acceleration is the time-rate at which the velocity of a moving body changes. Hence, acceleration is expressed as so many feet per second per second, so many miles per hour per minute, or any other convenient modifications of the units of distance and time.

For instance, if the velocity of a train increased uniformly from 30 to 40 miles per hour in 2 minutes, the acceleration might be expressed as 5 miles per hour per minute; 300 miles per hour per hour; $\frac{1}{12}$ of a mile per hour per second; 440 ft. per hour per second; $7\frac{1}{3}$ ft. per minute per second; or $\frac{11}{90}$ ft. per second per second. All these expressions are permissible; and each might be particularly suitable for some special purpose.

This change of motion might be stated (and, in practice, would probably be observed) in some such form as this: A train was found to travel 44 ft. in one second. The velocity was known to be increasing, and the acceleration was sensibly uniform. A second observation was made, half a minute after the first, when it was found that the train travelled 47·67 ft. in one second. A third observation, made one minute after the first, showed the train to travel 51·33 ft. in a second; and a fourth observation, two minutes after the first, showed the distance travelled in one second as 58·66 ft.

Clearly, then, in the last second, the train travelled farther than in the first by $58{\cdot}66 - 44 = 14{\cdot}66$ ft.; and since this increase of velocity was acquired in 2 minutes $=$ 120 seconds, the average acceleration was $14{\cdot}66$ ft. $\div 120 =$ $0{\cdot}122$ (or $\frac{11}{90}$) ft. per second per second.

It is important to notice that it follows from the foregoing consideration that acceleration is, in fact, a distance —*i.e.* real things accruing to a group or assemblage. The student will do well to analyse for himself the processes of subdividing the increase of distance into as many equal parts as there are seconds in the period occupied for the

accumulation of the increase, and the allocation of one such part of the increment-distance to each second of the period. He should satisfy himself thus of the fact that acceleration is really a simple thing, involving nothing more mysterious than measurements with a tape (or, possibly, counting the revolutions of a wheel of known diameter) and a stop-watch—and certainly not calling for such feats as the division of distance by "square seconds."

Obviously, acceleration is the differential coefficient of velocity with respect to time—*i.e.* the *second* differential coefficient of distance with respect to time—concerning the motion of a moving body; and where the change of velocity is not uniform, the methods of differentiation must be employed to determine the acceleration.

Integrating acceleration with respect to time, velocity is determined; and integrating velocity, the distance travelled is determined—thus proving again that both velocity and acceleration are essentially *distance*; for otherwise, increments of them, simply added together by the process of integration, could not produce a distance.

Retardation is conveniently regarded as negative acceleration.

Force.—When an unbalanced force is applied to a stationary body, either that body moves, or we must provide an adequate resistance if motion is to be prevented. Hence, force is usually defined as that which produces, or tends to produce, motion of a body to which it is applied. If a body be already moving in a certain direction, and a force be applied to it in the opposite direction, its motion will be retarded; but this is simply due to the effect of the force in tending to produce motion in the direction opposite from that of the initial motion—which latter is the effect of some force previously applied to the body.

Our measurements of "force" are merely comparisons of the motion-producing effects or tendencies observable in bodies to which forces are applied; and the basis of all such measurement is that force which we call the weight

of a body, together with the earthward—or " gravitational " —acceleration.

A body near the surface of the earth will move, if not prevented from so doing, towards the centre of the earth, under the influence of the resultant attraction which draws it in that direction. It is important to notice that this attraction is not concerned with the body in question and the earth alone. The body is being attracted in a multitude of other directions simultaneously, by the sun, the moon, the stars, other bodies in its neighbourhood—in fact, by all other matter in its many and various forms. Some of these attractions act more or less in conjunction, others in opposition, and others between these two extremes; while there are changes in them according to position and time. For all practical purposes, however, we regard the resultant attraction (towards the centre of the earth) as the single gravitational attraction; and we call the force of that attraction upon any particular body the *weight* of that body.

It is well to imagine a body as made up of a large number of extremely small *particles*; and also to imagine each of these particles attached to a corresponding particle of the earth by means of a fine cord pulling it earthwards. The pull in each of these fine cords we may call *one particle-attraction*; and we may conveniently imagine all particle-attractions equal. It is the sum—*i.e.* the combined action —of all these particle-attractions which forms what we know as the weight of a body; and our commercial units of force are based upon comparisons of weight.

When we say that a wind is assumed to exert a pressure of 20 lb. per square foot upon a wall, we mean that the effect to be provided for is that which would be produced by a weight of 20 lb. hanging vertically from a cord which, passing over a pulley, is attached to one square foot of the wall, as indicated in Fig. 95; and a similar weight applied to every other square foot of the surface exposed to the action of wind pressures.

When a projectile is discharged from a gun, we may estimate the force of the expulsion from the velocity imparted to the projectile, by thinking of the force as though it were applied by a weight, after the manner of Fig. 95, but the weight coming to earth at the instant in which the projectile emerges from the muzzle of the gun—*i.e.* the weight first suspended at a height (above the ground) equal to the length of the gun-barrel. The force thus estimated would, of course, be less than that of the actual explosion; and this fact must be taken into account in

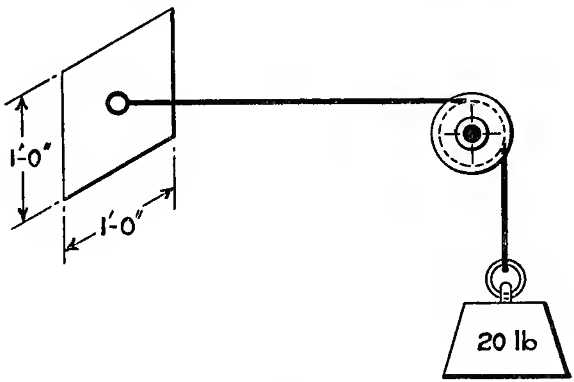

FIG. 95.

designing the gun for strength to resist the bursting pressures.

Obviously, we could not properly compare the magnitudes of forces by comparing the *velocities* which they impart to bodies; for a body of known weight, found to be moving with a certain velocity, might have acquired that velocity under the influence of a certain force acting for some particular period, or a force of half that magnitude acting for double that time, or any other magnitude of force acting for a suitably varied period. To be reliable, comparisons of forces must be based upon the *accelerations* which they impart to bodies upon which they act.

The earthward acceleration of a body is not affected by the weight of the body; and the reason for this will be clear when it is observed that, whether a body consist of two particles or two million particles, each particle-attraction will act by itself.

Consider two separate parcels of fine sand, each weighing 1 lb. Were it not for the disturbing effects of air resistances, the grains of either parcel would fall earthwards together if released simultaneously from some height, whether they were loose or held together by some bag or other container; for each individual grain is acted upon by its own particular set of particle-attractions. As regards this point of view, a solid body differs from a heap of granular material only in that the particles of the former are held together by cohesion; whereas in the latter the grains may be separated easily.

If the two parcels of sand were placed side by side, and allowed to fall together from some height, the acceleration of the whole would be the same as that of either by itself; for the particle-attractions of the one would still act upon the particles of that one, without being altered (appreciably) in any way by the approach or nearness of the other. The weight of the two together will, however, be twice that of either by itself; for the number of particle-attractions drawing the two parcels earthwards will be double of that operating upon one parcel alone.

Now let us consider the effects of applying a vertically downward force of 5 lb. to a body weighing 15 lb. The conditions may be realised by causing the earthward attraction of a 5-lb. weight to act vertically upwards upon a 10-lb. weight, as indicated in Fig. 96. The two bodies are supposed to be solid—*i.e.* the particles of each held together by cohesion—and hence, all the particle-attractions of the 5-lb. weight may be harnessed to the particles of the 10-lb. weight by a single cord, as shown. Were the two bodies granular, however, each grain of the 5-lb. weight might—except for merely mechanical difficulties—be

attached to an equal grain of the 10-lb. weight by its own separate cord passing over its own particular pulley; and it is a simple matter to extend this principle, in imagination, to the two weights of Fig. 96, particle by particle, even though they be solid bodies. Clearly, if this were done, and the particles of each body were quite free (instead of being held together by cohesion), only one-half of the particles of the 10-lb. weight would have cords attached to them, leaving the remainder free to break away and fall earthwards under the influence of their gravitational attraction. Under such conditions these particles would, of course, acquire an acceleration of 32·2 feet per second per second; and the sum of the particle-attractions so acting would be 10 lb. − 5 lb. = 5 lb. Since the bodies are solid, however, the net result will be that these 5 lb. of particle-attractions will operate upon the whole 15 lb. of particles to cause movement with acceleration in the direction indicated by the facts of Fig. 96, the 10-lb. weight moving earthwards, and the 5-lb. weight upwards. It would require 15 lb. of particle-attractions to impart an acceleration g to the whole assemblage of particles; and hence, the acceleration actually imparted will be

$$a = g\left(\frac{5\text{ lb.}}{15\text{ lb.}}\right) = \frac{g}{3} = \frac{32\cdot 2}{3}$$
$$= 10\cdot 73 \text{ feet per second per second,}$$

the cohesion and cord having the effect of harnessing each unbalanced particle-attraction to three particles.

The student should apply similar argument to modifications of the case considered above—for instance, replacing the 5-lb. weight by one of 4 lb., and the 10-lb. by one of 3 lb., giving an acceleration

$$a = g\left(\frac{3-4}{3+4}\right) = -g\left(\frac{1}{7}\right)$$
$$= -4\cdot 6 \text{ feet per second per second,}$$

the negative sign indicating that the motion is in the direction opposite from that of Fig. 96.

Generally, if the weights be W_1 lb. and W_2 lb., as indicated in Fig. 97, the unbalanced earthward particle-attractions will amount to $(W_1 - W_2)$ lb.; and the motion-producing effects of these will be harnessed to the whole assemblage of particles represented by $(W_1 + W_2)$. Since $(W_1 + W_2)$ lb. of particle-attractions would be required to impart to the whole assemblage of particles an acceleration

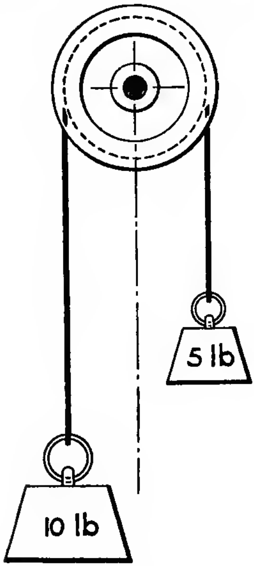

FIG. 96.

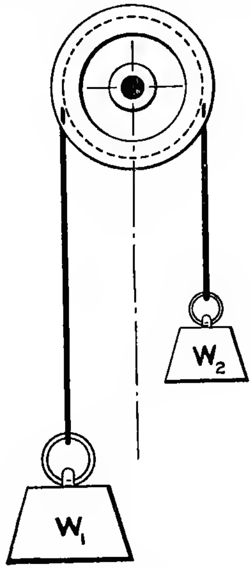

FIG. 97.

of g, it follows that the acceleration actually imparted will be

$$a = g\left(\frac{W_1 - W_2}{W_1 + W_2}\right).$$

This, of course, ignores the weight of the cord, and also that of the pulley (the motion of which would occupy some of the particle-attractions, and thus reduce the acceleration a), as well as frictional and other resistances.

Next, consider the case of a solid body—*e.g.* a car—

weighing in all 100 lb., mounted upon frictionless wheels, and standing upon a truly horizontal plane (as indicated in Fig. 98) in a vacuum—so that, while earthward movement is entirely prevented, there is no resistance to horizontal motion.

Suppose a horizontal force of 1 lb. be applied to the body, by means of a weight and cord, as in Fig. 98. The body will move horizontally, of course; and so long as the 1-lb. weight is attached to it, the acceleration imparted will be

$$a = g\left(\frac{1}{100+1}\right) = \frac{32 \cdot 2}{101}$$
$$= 0 \cdot 32 \text{ feet per second per second.}$$

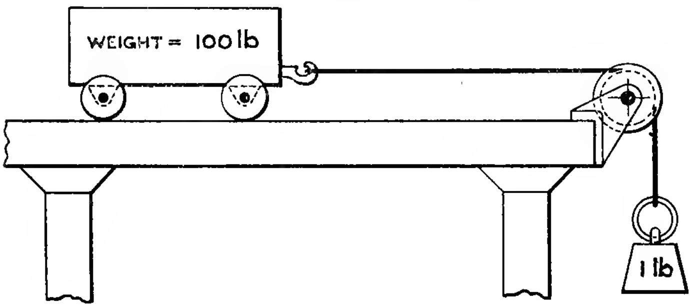

FIG. 98.

The particle-attractions necessary to impart an acceleration of 32·2 feet per second per second to the car would amount to 100 lb.; and as there are available only so many as amount to 1 lb., acting directly upon the weight, each of these is harnessed, by cohesion, etc., to 100 others as well, reducing its accelerating effect upon the whole in the ratio of 1 : 101.

Similarly, if it be desired to impart an acceleration of, say, 2 feet per second per second horizontally to the 100-lb. car of Fig. 98, the force necessary will be

$$F = W\left(\frac{a}{g}\right) = 100 \text{ lb.}\left(\frac{2}{32 \cdot 2}\right)$$
$$= 6 \cdot 21 \text{ lb.}$$

If, however, the horizontal force is to be applied by means of a weight, as in Fig. 98, such weight must be more than 6·21 lb., because that body also must be accelerated in order to keep it constantly applied to the car. The magnitude of the force necessary might be determined thus:

$$F = (100 + F)\left(\frac{2}{32 \cdot 2}\right) = \frac{100 + F}{16 \cdot 1},$$

whence

$$16 \cdot 1F = 100 + F$$

$$F(16 \cdot 1 - 1) = 100$$

$$F = \frac{100}{15 \cdot 1} = 6 \cdot 62 \text{ lb.}$$

The general inference may, then, be expressed:

$$F : W :: a : g;$$

whence

$$F = W\left(\frac{a}{g}\right)$$

and

$$a = g\left(\frac{F}{W}\right).$$

Or, in words:

> Where a known acceleration is imparted to a given body, the force applied to that body (in the direction of the observed acceleration) is equal to the weight of the body, varied in the ratio borne by the observed acceleration to the gravitational acceleration;

and

> Where a force of known magnitude is applied to a given body, the acceleration imparted to that body (in the direction of the applied force) is equal to the gravitational acceleration, varied in the ratio borne by the known force to the weight of the body.

Obviously, so long as a force acts upon a body, the acceleration due to that force will continue to be imparted to the body, its velocity increasing continually. If the

force be withdrawn after an application extending over some period, the body will continue to move in the direction, and with the velocity, of its motion at the instant of such withdrawal. In actual working, of course, the motion would very soon be altered, in magnitude or direction (or both) by the resistances encountered by the body, or by the application to it of other forces. Were it not for such effects, the motion would presumably continue, with unchanged direction and constant velocity, indefinitely.

Mass.—It will be noticed that the foregoing discussion of force contains no reference to "mass" (see Chapter VI.); and that, for the purposes of practical calculation and investigation, mass is unnecessary. Many books contain very definite references to mass, however; and as one of the principal objects here is to assist students in their reading, it will be well to consider the matter a little more thoroughly in the light of the foregoing argument.

Two sets of things were dealt with in our consideration of the earthward attraction of a body—(1) *particles* of the body; and (2) *particle-attractions* which act upon those particles.

Now, a particle of a body is, clearly, distinct and different from the particle-attraction which acts upon it; and this, doubtless, is the reason for the introduction of the term and conception of "mass."

With a 1-lb. weight falling freely earthwards, we have (so to speak) 1 lb. of particle-attractions acting upon 1 lb. of particles; but, as shown above, we may have 1 lb. (or any other quantity) of particle-attractions applied to 100 lb. (or any other quantity) of particles. Physicists object to the "pound" (which is a weight-unit) being applied indiscriminately to both; and hence, it would appear, an endeavour was made to deduce a unit for the *quantity of particles*, such unit being termed a unit of *mass*, or unit *quantity of matter*.

Obviously, a clear definition of a "particle" or "unit of mass" would be convenient—if we really dealt with

particles or mass as such; but since we deal, in fact, only with motion-producing tendencies, and call these "weights," it is not surprising to find that, even when a unit of mass is deduced, it is really a weight. Physicists often speak of a mass of so many *pounds* being acted upon by a force of so many *pounds*; and this is open to the selfsame objection as is the statement at the opening of the last preceding paragraph.

As regards basic principle, the argument commonly applied in the deduction of a unit of mass is as follows:

The gravitational acceleration is 32·2 feet per second per second—*i.e.* an assemblage comprising 1 foot per second per second repeated to 32·2 times. Similarly the acceleration a feet per second per second may be regarded as $a \times 1$ foot per second per second.

Hence, the relation $F = W\left(\frac{a}{g}\right)$ might be stated as

$$F = W \text{ lb.}\left(\frac{a \times 1 \text{ ft. per sec. per sec.}}{32{\cdot}2 \times 1 \text{ ft. per sec. per sec.}}\right);$$

and, dividing the weight W into as many equal parts as there are repetitions of the unit acceleration in g,

$$\begin{aligned} F &= \frac{W \text{ lb.}}{32{\cdot}2}\left(\frac{a \times 1 \text{ ft. per sec. per sec.}}{1 \text{ ft. per sec. per sec.}}\right) \\ &= \frac{W \text{ lb.}}{32{\cdot}2}(a). \end{aligned}$$

It is necessary to note that, in consequence of the argument adopted, a is no longer an acceleration, but merely a factor of repetition.

If $W = 1$ lb. and $a = 1$, the relation would become

$$\begin{aligned} F &= \frac{1 \text{ lb.}}{32{\cdot}2} \\ &= \text{(practically) half an ounce}; \end{aligned}$$

which means neither more nor less than that, if half an ounce of particle-attractions be applied to 1 lb. of particles,

the acceleration imparted by the former to the latter will be 1 foot per second per second.

The definitions given in good text-books on Physics and Mechanics should be examined carefully on this basis; when it will be found that their real significance is as stated above, whatever be the wording in which they are expressed or described.

Mostly, of course, Physicists adopt the "pound" as the unit of *mass*; and apply the foregoing argument to determine the unit of *force*, as shown. To the "unit" force of 1 lb. $\div$ 32·2 (*i.e.* practically half an ounce) the term "poundal" is applied; but it is open to question as to whether this has ever served any real practical purpose.

The alternative method might be followed, of course, the "pound" being adopted as the unit of *force*; when the unit of *mass* would be 32·2 lb.—for which, were it worth while, a special name could easily be invented; though such treatment would not alter the fact that it is very slightly more than $1\frac{1}{7}$ qr., or $\frac{2}{7}$ cwt.

Work.—When a force overcomes a resistance through a distance, something has happened which we cannot measure by comparison with any unit of weight or length alone. We describe the happening by saying that "work has been done"; and a unit of "work" in common acceptance is *that amount of work done when a resistance of* 1 *lb. is overcome through a distance of* 1 *foot.*

The student should satisfy himself that in measuring work, we do *not* multiply force by distance. If a resistance of 7 lb. were overcome through a distance of 4 feet, all that we need to—and all that we can—say is that, since the resistance is 1 lb. repeated to 7 times, 7 units of work will be done in overcoming this resistance through 1 foot; and since it is overcome through 1 foot repeated to 4 times, there will be 4 groups of work-units done, each group comprising 7 units, whence the total assemblage will comprise 28 work-units—*i.e.* 28 ft.-lb., or 1 ft.-lb. repeated to 28 times.

Moment.—A body will rotate if subjected to the action of an unbalanced couple; and such rotational tendency is called "moment." Evidently, rotational tendencies cannot be measured by reference to units of length or of weight alone, though the cause of such a tendency is a couple comprising two forces at a leverage.

One unit in common acceptance is *that amount of rotational tendency set up by a couple comprising forces of* 1 *lb. each with a leverage of* 1 *foot.* This unit is sometimes called a "foot-pound"; and sometimes (to distinguish it from the unit of work) a "pound-foot." Seeing that both are mere names, and that units of work are seldom involved in calculations relating to moments, the student may

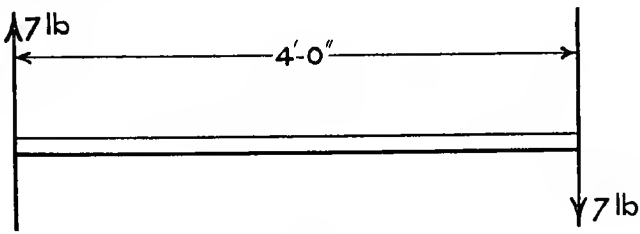

FIG. 99.

please himself—in his practical work, though perhaps not in examinations—as to whether he uses the same title for both or a distinct title for each.

If a couple comprising forces of 7 lb., with a leverage of 4 ft., act alone upon a body, as indicated in Fig. 99, the rotational tendency imparted to the body will be 28 ft.-lb.; but this is not the result of multiplying 7 lb. by 4 ft.

Each force of 7 lb. is really 1 lb. repeated to 7 times; and since the same rotational tendency could be produced by repeating the forces to four times if the leverage be reduced to one-fourth of its former magnitude, the given couple might be replaced by the equivalent couple comprising forces of 28 lb. with a leverage of 1 ft., as indicated in Fig. 100.

Now, a force of 28 lb. is merely a force of 1 lb. repeated to 28 times; and hence Fig. 100 might be the side view of 28 separate couples, each comprising forces of 1 lb. at a leverage of 1 ft., acting side by side. Each of these 28 couples will impart one unit of rotational tendency to the body upon which they act; and it follows, therefore, that the rotational tendency of the couple indicated in Fig. 99 is 28 units—*i.e.* 28 ft.-lb., or 28 lb.-ft.

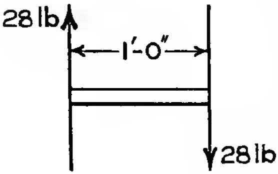

FIG. 100.

It is important to notice—(1) that a moment is the rotational tendency imparted *to a body*; and (2) that such a tendency can only be imparted by a *couple*. The statements commonly made concerning the *moment of a force about a point* should be used with the most particular care (at least in thought) to avoid looseness or misunderstanding.

Given a force F and a point P, as in Fig. 101, the force would have no effect whatever upon the point (even regarding the latter as a small body); for the force is not applied to the point. Further, even were the point P a particle in a body such as PQ upon which the force F might act, as in Fig. 102, the effect of the single force would be only to move PQ bodily in the direction of F. There could be no tendency for PQ to rotate about P unless movement of P were prevented by the application to it of a *reaction* R = F, as shown in Fig. 103.

Many of the couples in practice comprise a force and a reaction, instead of two forces; and it is therefore convenient to notice the essential difference between force and reaction. The former is *active*, and can follow a body to which it has imparted motion without loss of effectiveness. The latter is *passive*, and can only oppose or prevent the motion which an applied force would other-

wise impart. Its function is to *prevent* motion, and it has no power to *impart* motion.

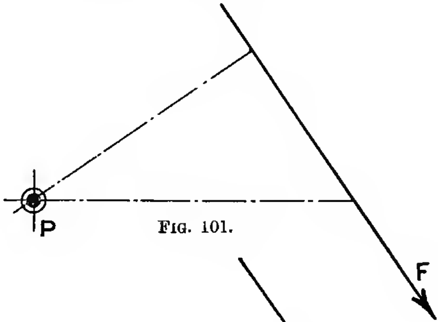

FIG. 101.

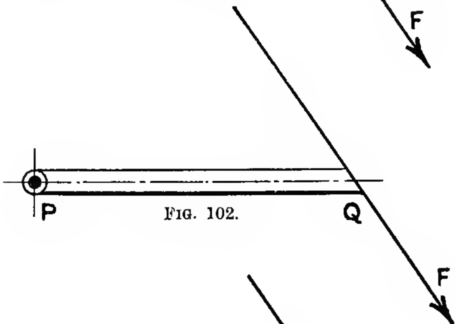

FIG. 102.

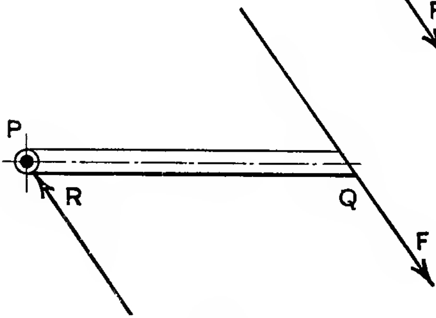

FIG. 103.

The consequences of this are well worth careful consideration, for they may be the means of simplifying practical calculations in many ways.

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